$\Omega \cong \diamond\mathcal{W}$

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1 Definition

1.1 Ω

Eventually every correct process p satisfies $\Omega_p = c$ **forever**, where c is a correct process.

1.2 $\diamond\mathcal{W}$

Eventual Weak Accuracy + Weak Completeness :

Eventual weak accuracy: *There is a time after which some* correct process is never suspected by any correct process.

Weak Completeness: Every faulty process is eventually permanently suspected by *some* correct process.

2 System Assumptions

1. Complete network topology.
2. Asynchronous identified processes: a process and its identifier are used equivalently.
3. Asynchronous reliable links (not necessarily FIFO).

We consider executions as sequences of (atomic) events $(e_i)_{i\geq 0}$ where each event e_i occurs at a given process during the time interval from time i to time $i+1$. An event may be:

- an internal event, corresponding to a local computation performed by the process;
- a send event, in which the process sends a message;
- or a receive event, in which the process receives a message.

3 Notations

- V: the set of processes
- $Correct \subseteq V$: the set of correct processes.
- $Faulty \subseteq V$: the set of faulty processes.
- X_p: the value of variable X of process p.
- X_p^t: the value of variable X of process p at time t.

4 $T_{\Omega\to\diamond\mathcal{W}}$

Algorithm 1 $T_{\Omega\to\diamond\mathcal{W}}$, code for every process p

```
Function T_{Ω→◇W}(p)
    return V \ {Ω_p}
End Function
```

Question 1. Prove that $T_{\Omega\to\diamond\mathcal{W}}$ satisfies weak completeness.

Actually, $T_{\Omega\to\diamond\mathcal{W}}$ satisfies strong completeness (as shown below). Now, strong completeness implies weak completeness.

By definition, $V = \mathit{Faulty} \,\dot{\cup}\, \mathit{Correct}$.

Let p be any correct process. By definition of Ω, there is a time after which $\Omega_p \in \mathit{Correct}$ forever. So, there is a time after $\{\Omega_p\} \subseteq \mathit{Correct}$. Hence, there is a time after which $\mathit{Faulty} \subseteq V \setminus \{\Omega_p\}$, *i.e.*, there is a time after which $\mathit{Faulty} \subseteq T_{\Omega\to\diamond\mathcal{W}}(p)$. $\square$

Question 2. Prove that $T_{\Omega\to\diamond\mathcal{W}}$ satisfies eventual weak accuracy.

By definition of Ω, there is a time after which $\Omega_p = c$ for every correct process p where $c \in \mathit{Correct}$.

Hence, by definition of the algorithm, there is a time after which $c \notin T_{\Omega\to\diamond\mathcal{W}}(p)$ for every correct process p. In other words, there is a time after which the correct process c is never more suspected by every correct process p. $\square$

5 $T_{\diamond\mathcal{W}\to\Omega}$

Question 1. Following the principles presented in the lesson, propose an algorithm $T_{\diamond\mathcal{W}\to\Omega}$.

Notations: Please use the following variables:

- *Leader* $\in V$, initialized to $\min V$.
- $C[]$, array of integers indexed on V, every cell is initialized to 0.

Algorithm 2 $T_{\diamond\mathcal{W}\to\Omega}$, code for every process p

Variables:
    Leader $\in V$, initialized to $\min V$
    $C[]$, array of integers indexed on V, every cell is initialized to 0
End Variables

Function $T_{\diamond\mathcal{W}\to\Omega}(p)$
    return *Leader*
End Function

while *true* **do**   */* Must be run into a separated thread */*
    `broadcast` $\langle C, p\rangle$ `to` $V \setminus \{p\}$
    For all $q \in V \setminus \{p\}$ **do**
        If `receive` $\langle CN, q\rangle$ **then**
            For all $x \in V$ **do**
                $C[x] \leftarrow \max(C[x], CN[x])$
            End For
        End If
    End For
    For all $q \in \diamond\mathcal{W}(p)$ **do**
        $C[q]++$
    End For
    Leader $\leftarrow \min\{q \in V \mid \forall q' \in V, C[q] \leq C[q']\}$
end while

Question 2. Show the following lemma.

Lemma 1. *$\forall p \in Correct$, $\forall q \in Faulty$, $\forall t \in \mathbb{N}$, $\exists t' > t$ such that $C[q]_p^t < C[q]_p^{t'}$.*

Proof. First, if $Correct = \emptyset$, then the lemma trivially holds. Assume otherwise that $Correct \neq \emptyset$ and let $p \in Correct$. By definition of the algorithm, $C[q]_p$ is monotonically nondecreasing with time, *i.e.*, $C[q]_p^t \leq C[q]_p^{t'}$, $\forall t, t' \in \mathbb{N}$ with $t' > t$.

Then, since $\diamond\mathcal{W}$ satisfies weak completeness, there is a correct process r and a time t_0 such that $\forall t \geq t_0$, $q \in \diamond\mathcal{W}(r)^t$.

Since r is correct, r executes $C[q]++$ infinitely often, *i.e.*, $C[q]_r$ regularly increases. Moreover, r broadcasts C_r infinitely often and, as the links a reliable, every correct process $p \neq r$ receives C_r infinitely often. By Line 13, $C[q]_p$ also regularly increases and we are done. □

Question 3. Show the following lemma.

Lemma 2. *If $Correct \neq \emptyset$, then $\exists p \in Correct$ and $\exists k, t \in \mathbb{N}$ such that $\forall q \in Correct$, $\forall t' \geq t$, $C[p]_q^{t'} \leq k$.*

Proof. Assume $Correct \neq \emptyset$. Since $\diamond\mathcal{W}$ satisfies eventual weak accuracy, $\exists \ell \in Correct$, $\exists t_\alpha \in \mathbb{N}$ such that $\forall p \in Correct$, $\forall t' \geq t_\alpha$, $\ell \notin \diamond\mathcal{W}(p)$.

Let t_β be the time from which every faulty process has crashed and every message sent by a faulty process to a correct process has been received. Let $t_0 = \max(t_\alpha, t_\beta)$. Let $Max = \max_{p \in Correct} C_p^{t_0}[\ell]$.

We now prove by induction on t that $\forall t \geq t_0$, $\forall p \in Correct$, $C_p^t[\ell] \leq Max$ and every message $\langle CN, \_\rangle$ in transit to p at time t satisfies $CN[\ell] \leq Max$.

Base Case: $t = t_0$. By Definition, $\forall p \in Correct$, $C_p^{t_0}[\ell] \leq Max$. Moreover, since $C_p[\ell]$ is monotonically nondecreasing with time and no message in transit at time t_0 has been sent by a faulty process (by definition of t_0), every message $\langle CN, \_\rangle$ in transit to p at time t_0 satisfies $CN[\ell] \leq Max$.

Induction Step: Consider the step from time $t \geq t_0$ to time $t + 1$. Let $p \in Correct$.

By definition p and t_0, never more execute Line 18 to modify $C_p[\ell]$.

So, $C_p[\ell]$ can only be modified by Line 13: if p is executed Line 13 during the step from time t to time $t + 1$, $C_p[\ell]$ is assigned to the maximum value between $C_p[\ell]^t$ and the value $CN[\ell]$ of a message received by time t.

Now, by induction hypothesis, $C_p[\ell]^t$ and all these received values $CN[\ell]$ are less than or equal to Max.

Hence, $C_p^{t+1}[\ell] \leq Max$.

Finally, every message in transit at time $t+1$ has been sent by time $t+1$. Moreover, from the algorithm, every message sent by time $t + 1$ contains values of arrays C modified not later than time t. Hence, from the induction hypothesis and owing the fact that only correct processes can send messages during the step from time t to time $t + 1$, we can deduce that every message $\langle CN, \_\rangle$ in transit to any correct process at time $t + 1$ satisfies $CN[\ell] \leq Max$.

Thus, the induction holds at time $t + 1$ and we are done.

Hence, the lemma holds with $p = \ell$. □

Question 4. Show the following lemma.

Lemma 3. $\exists t \in \mathbb{N}$ *such that* $\forall p \in Correct$, $\forall t' \geq t$, $C_p^t[Leader_p^t] = C_p^{t'}[Leader_p^{t'}]$.

Proof. Assume, by the contradiction, that $\forall t \in \mathbb{N}$, $\exists p \in Correct$ and $\exists t' > t$ such that $C_p^t[Leader_p^t] \neq C_p^{t'}[Leader_p^{t'}]$.

First, since the number of correct process is finite, $\exists p \in Correct$ such that $\forall t \in \mathbb{N}$, $\exists t' > t$ such that $C_p^t[Leader_p^t] \neq C_p^{t'}[Leader_p^{t'}]$.

Moreover, since $C_p[q]$ is monotonically nondecreasing for all $q \in V$, we have: $\exists p \in Correct$ such that $\forall t \in \mathbb{N}$, $\exists t' > t$ such that $C_p^t[Leader_p^t] < C_p^{t'}[Leader_p^{t'}]$.

By Line 20 and owing the fact that V is finite, we have $\forall q \in V$, $\forall t \in \mathbb{N}$, $\exists t' > t$ such that $C_p^t[q] < C_p^{t'}[q]$, a contradiction to Lemma 2. □

Question 5. Show the following corollary.

Corollary 1. $\exists t \in \mathbb{N}$ *such that* $\forall p \in Correct$, $\forall t' \geq t$, $Leader_p^t = Leader_p^{t'}$.

Proof. Assume, by the contradiction that $\forall t \in \mathbb{N}$, $\exists p \in Correct$, $\exists t' > t$ such that $Leader_p^t \neq Leader_p^{t'}$.

Since the number of correct processes is finite, we have:

Claim 1: $\exists \ell \in Correct$ such that $\forall t \in \mathbb{N}$, $\exists t' > t$ such that $Leader_\ell^t \neq Leader_\ell^{t'}$.

Let $t_0 \in \mathbb{N}$ such that $\forall p \in Correct$, $\forall t' \geq t_0$, $C_p^{t_0}[Leader_p^{t_0}] = C_p^{t'}[Leader_p^{t'}]$. By Lemma 3, t_0 is well-defined. Let $n = |V|$. By Claim 1, $\exists t_1, t_2, \ldots, t_n$ such that $\forall i \in [1..n]$, $t_i > t_{i-1}$ and $Leader_\ell^{t_i} \neq Leader_\ell^{t_{i-1}}$. So, $\exists i, j \in [0..n]$ such that $i < j$ and $Leader_\ell^{t_i} = Leader_\ell^{t_j}$. By definition of t_0, $C_\ell^{t_i}[Leader_\ell^{t_i}] = C_\ell^{t_0}[Leader_\ell^{t_0}] = C_\ell^{t_j}[Leader_\ell^{t_j}]$. Now, by Line 20 and owing the fact that counters are monotonically nondecreasing, $C_\ell^{t_i}[Leader_\ell^{t_i}] < C_\ell^{t_{i+1}}[Leader_\ell^{t_i}] \leq C_\ell^{t_j}[Leader_\ell^{t_i}] = C_\ell^{t_j}[Leader_\ell^{t_j}]$, a contradiction. □

Question 6. Show the following lemma.

Lemma 4. $\exists t \in \mathbb{N}$ *such that* $\forall t' \geq t$, $\forall p \in Correct$, $Leader_p^{t'} = \ell$ *where* $\ell \in Correct$.

Proof. Let $t_z \in \mathbb{N}$ such that $\forall p \in Correct$, $\forall t' \geq t_z$, $Leader_p^{t_z} = Leader_p^{t'} \wedge C_p^{t_z}[Leader_p^{t_z}] = C_p^{t'}[Leader_p^{t'}]$. By Lemma 3 and Corollary 1, t_z is well-defined.

By Lemma 1, $\forall p \in Correct$, $Leader_p^{t_z} \in Correct$.

Assume now, by the contradiction, that $\exists p_1, p_2 \in Correct$ such that $Leader_{p_1}^{t_z} \neq Leader_{p_2}^{t_z}$.

Without the loss of generality, assume that $C_{p_1}^{t_z}[Leader_{p_1}^{t_z}] < C_{p_2}^{t_z}[Leader_{p_2}^{t_z}] \vee (C_{p_1}^{t_z}[Leader_{p_1}^{t_z}] = C_{p_2}^{t_z}[Leader_{p_2}^{t_z}] \wedge Leader_{p_1}^{t_z} < Leader_{p_2}^{t_z})$. This, in particular, means that $C_{p_2}^{t_z}[Leader_{p_2}^{t_z}] \geq C_{p_1}^{t_z}[Leader_{p_1}^{t_z}]$.

By definition of the algorithm, $C_{p_2}^{t_z}[Leader_{p_1}^{t_z}] \geq C_{p_2}^{t_z}[Leader_{p_2}^{t_z}]$.

- Assume that $C_{p_2}^{t_z}[Leader_{p_1}^{t_z}] = C_{p_2}^{t_z}[Leader_{p_2}^{t_z}]$. Then, $Leader_{p_2}^{t_z} \leq Leader_{p_1}^{t_z}$.
  So, $C_{p_1}^{t_z}[Leader_{p_1}^{t_z}] < C_{p_2}^{t_z}[Leader_{p_2}^{t_z}]$, *i.e.*, $C_{p_2}^{t_z}[Leader_{p_1}^{t_z}] > C_{p_1}^{t_z}[Leader_{p_1}^{t_z}]$.
- Otherwise, $C_{p_2}^{t_z}[Leader_{p_1}^{t_z}] > C_{p_2}^{t_z}[Leader_{p_2}^{t_z}] \geq C_{p_1}^{t_z}[Leader_{p_1}^{t_z}]$.

Hence, in any case, $C_{p_2}^{t_z}[Leader_{p_1}^{t_z}] > C_{p_1}^{t_z}[Leader_{p_1}^{t_z}]$.

Since p_2 is correct and the counters are monotonically nondecreasing, p_2 sends infinitely many $\langle CN, p_2 \rangle$ messages to p_1 with $CN[Leader_{p_1}^{t_z}] > C_{p_1}^{t_z}[Leader_{p_1}^{t_z}]$. Since the link are reliable and p_1 is correct, by Line 13, $\exists t' > t_z$ such that $C_{p_1}^{t_z}[Leader_{p_1}^{t_z}] < C_{p_1}^{t'}[Leader_{p_1}^{t_z}]$, a contradiction.

Hence, $\forall p, q \in Correct$, $Leader_p^{t_z} = Leader_q^{t_z}$, and with $t = t_z$, the lemma holds. □