

Hierarchical Routing

Réseaux & Communication

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Roadmap

- 1 Introduction
- 2 Examples of clustering
 - Clustering of [3]
 - Clustering of [1]
- 3 Routing in a Clustering
- 4 References

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To reduce the **cost parameters** of the routing:

use of a **hierarchical division** of the network

Justification: Most of the communication is **local**, *i.e.*, between nodes at “relatively” small distances from each other

① Length of addresses

n nodes $\Rightarrow$ at least $\log(n)$ bits per address

Maybe more, if information is encoded in addresses

E.g., the prefix routing.

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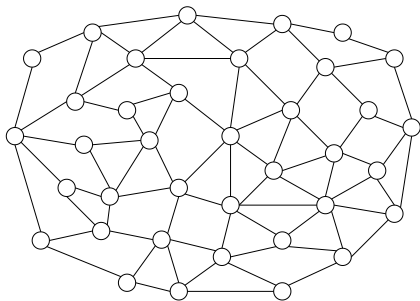
③ Cost of table lookups

The cost (in time) of a single table lookups is likely as larger for a large routing table or for larger addresses

The total table-lookup time for the delivery of a single message also depends on the number of times the tables must be accessed (number of hops)

Clustering

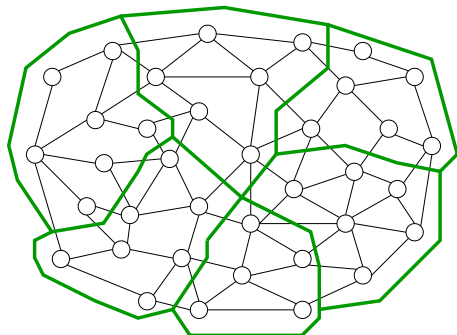
Definition



Clustering

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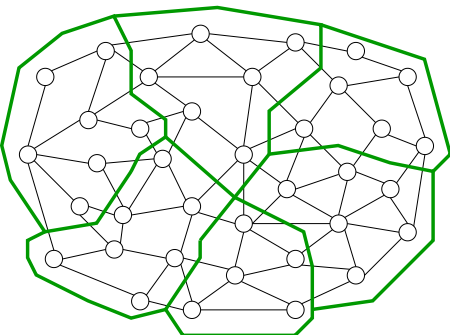
- Partition of the network into connected subgraphs called **clusters**



Clustering

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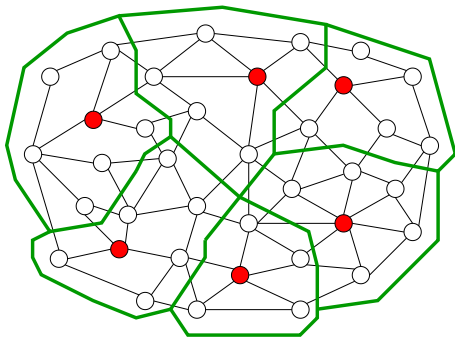
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- Partition of the network into connected subgraphs called **clusters**
- Each cluster is roughly of the same size
- In each cluster, there is a designated node: the **clusterhead**



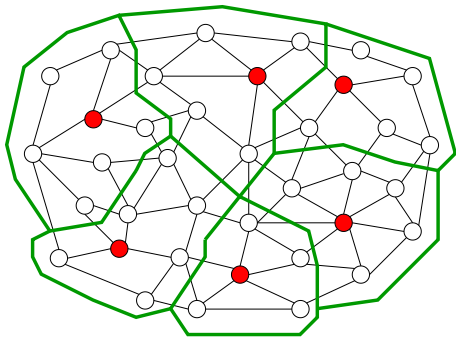
Clustering

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- Partition of the network into **connected subgraphs** called **clusters**
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Remark: Clustering **maybe recursive**, *i.e.*, each cluster may be partitioned into subclusters, and so on so forth, in order to obtain a multi-level division of the nodes.

Here, we only consider **1-level division**.

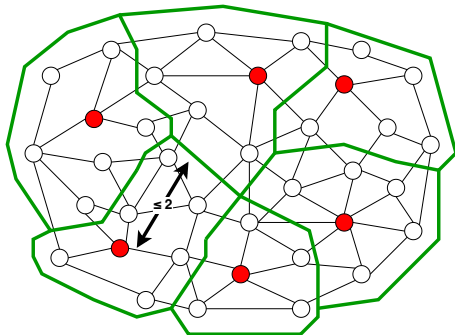


k-Clustering

A **k-clustering** is a clustering of the network where each cluster is of **radius at most k**

($k = 1 \Rightarrow$ dominating set)

In a cluster, each node is **at distance at most k from its clusterhead**.



2-clustering

A good k -Clustering?

Trivial solution: all nodes are clusterheads!

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Minimal k -clustering (e.g., [2]).

Let C be the set of clusterheads of a minimal k -clustering.

There is no k -clustering whose set of clusterheads is a proper subset of C .

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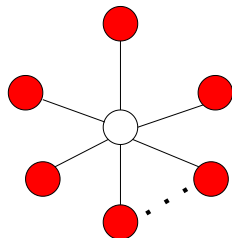
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minimal 1-clustering with $n - 1$
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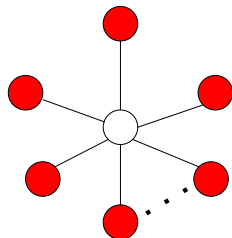
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At most M times the size of the optimal one.



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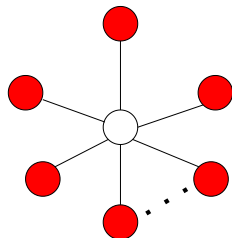
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Remark: a $(O(\frac{n}{k}))$ k -clustering can be made minimal to even more reduce the number of clusters [2].



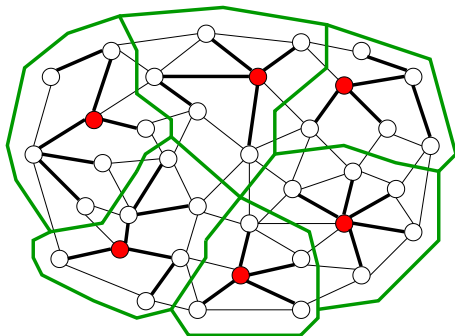
minimal 1-clustering with $n - 1$
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Clustering

(Usual) Structure

Cluster = Tree rooted at its
clusterhead

Spanning Forest



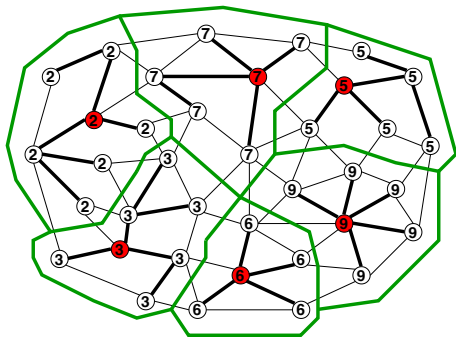
Clustering

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Spanning Forest

Colored Trees, e.g., with the clusterhead identifiers



Clustering-based Routing

A major application of *k*-clustering is in the implementation of an efficient routing scheme in a network.

To route a packet from p to q :

- 1 Route the packet from p to its clusterhead (intra-clustering)
- 2 Route the packet from the clusterhead of p to the clusterhead of q (inter-clustering)
- 3 Route the packet from the clusterhead of q to q (intra-clustering)

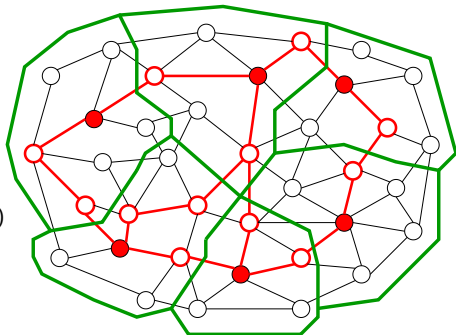
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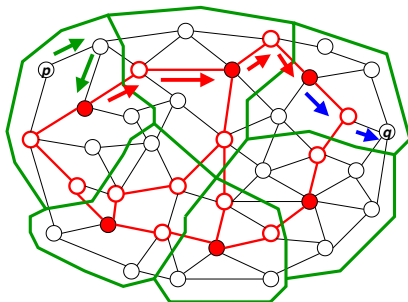
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Each of the three phases can be identified using a color

- Depending of its color, a packet is either forwarded over a fixed channel or a more complex routing scheme
- Each phase can be handled using a different protocol



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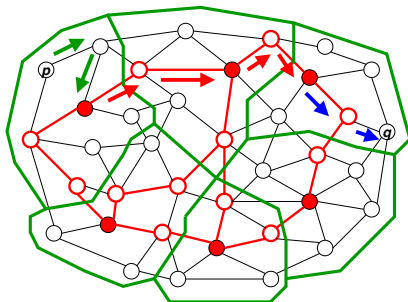
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Address: (Label of the clusterhead of the destination, destination label)



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Two examples

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Both computes $O(\frac{n}{k})$ clusters

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The second one also provides an approximation of the optimal clustering in case the network is a Unit (or Quasi Unit) Disk Graph

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Let $G = (V, E)$ be a connected graph of n nodes

A **k -dominating set** of G is a subset D of V such that $\forall p \in V, \exists q \in D$ such that $\|p, q\| \leq k$

k -dominating set $\approx$ set of clusterheads

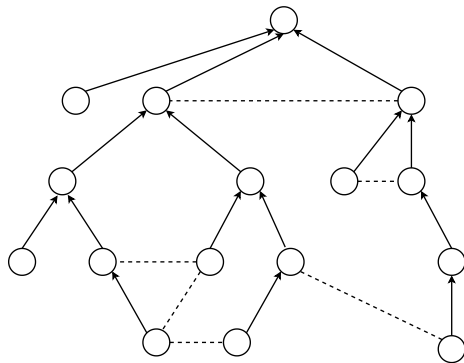
Theorem

For every $k \in \mathbb{N}$, there exists a k -dominating set D of G such that $|D| \leq \lceil \frac{n}{k+1} \rceil$

Proof of the Theorem

Some notations

Let $T = (V, E_T)$ be an arbitrary spanning tree of G rooted at node r

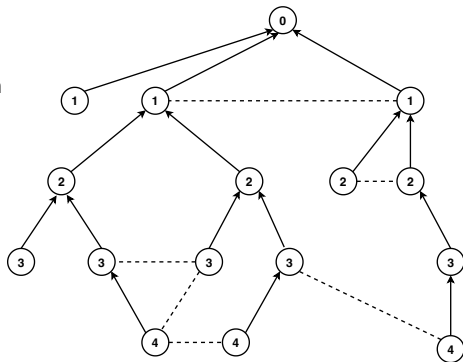


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Let $L(p) = \|p, r\|_T$ be the **level** of $p \in V$ in T



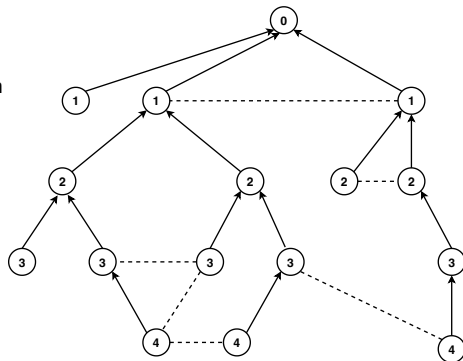
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Let $H = \max_{p \in V} L(p)$ be the **height** of T
(4 in the example)



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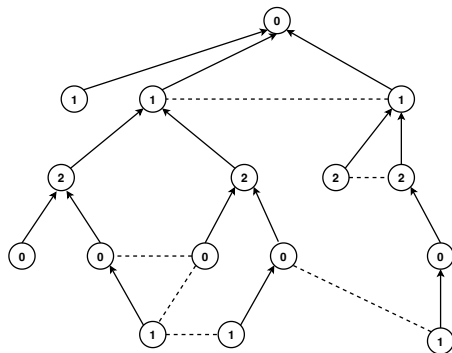
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For every $i \in [0..k]$, let

$$D_i = \{p \in V \mid L(p) \bmod (k+1) = i\}$$

(in the example, $k = 2$)



Proof of the Theorem

Easy cases: $n = 0$ or $k = 0$

- ① If $n = 0$, then $\lceil \frac{n}{k+1} \rceil = 0 = |\emptyset|$ and $\emptyset$ is a k -dominating set of G .
- ② If $k = 0$, then $D_0 = V$, and so D_0 is a k -dominating set of G .
Moreover, $|D_0| = n = \lceil \frac{n}{k+1} \rceil$.

Proof of the Theorem

Case $n > 0$ and $k > 0$

- ③ Assume $k \geq H$. Then, D_0 only contains r and every other node is within distance k from r . So, D_0 is a k -dominating set of G whose size is $1 \leq \lceil \frac{n}{k+1} \rceil$.

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⑥ Assume $k < H$. Then, $\forall i \in [0..k], |D_i| > 0$.

① Assume that $\forall i \in [0..k-1], |D_i| = |D_{i+1}|$.

Then, $\forall i \in [0..k], |D_i| = \lceil \frac{n}{k+1} \rceil$.

Let $v \notin D_0$. The level of v , $L(v)$, satisfies $L(v) = x(k+1) + y$, where $x \geq 0$ and $0 < y \leq k$.

Let u be the ancestor of v such that $L(u) = L(v) - y$ (u exists because $y \leq L(v)$).

By definition, $u \in D_0$ and $\|u, v\| \leq k$. Hence, D_0 is a k -dominating set of G such that $|D_0| = \lceil \frac{n}{k+1} \rceil$.

Proof of the Theorem

Case $n > 0$ and $k > 0$

- ② Assume that $\exists i \in [0..k-1], |D_i| \neq |D_{i+1}|$.

Let $\min \in [0..k]$ such that $\forall i \in [0..k], |D_{\min}| \leq |D_i|$.

Then, $|D_{\min}| < \lceil \frac{n}{k+1} \rceil$.

Let $D = D_{\min} \cup \{r\}$. Then, $|D| \leq \lceil \frac{n}{k+1} \rceil$.

Let $v \notin D$.

- ① If $L(v) \leq k$, then v is at distance at most k from r and $r \in D$.
- ② If $L(v) > k$, then $L(v) = x(k+1) + y$ with $x > 0$, $0 \leq y \leq k$, and $y \neq \min$ (if $y = \min$, then $v \in D_{\min} \subseteq D$, a contradiction).

If $y > \min$, then let u be the ancestor of v such that $L(u) = x(k+1) + \min$. Now, $0 \leq L(v) - L(u) = y - \min \leq k$.

If $y < \min$, let u be the ancestor of v such that $L(u) = (x-1)(k+1) + \min$. Now, $0 \leq L(v) - L(u) = k+1 + y - \min \leq k$.

By definition, $u \in D$ (more precisely, $u \in D_{\min}$) and $\|u, v\| \leq k$.

Hence, D is a k -dominating set of G and $|D| \leq \lceil \frac{n}{k+1} \rceil$.



Let $mcd = \min\{|D_i| \mid i \in [0..k] \wedge D_i \neq \emptyset\}$: *mcd* is the minimum cardinal of a non-empty *D*-set.

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Let $x = \min\{i \mid i \in [0..k] \wedge |D_i| = mcd\}$: x the the smallest index of a D -set of size mcd .

$D_x \cup \{r\}$ is a k -dominating set of G if size at most $\lceil \frac{n}{k+1} \rceil$.

Proof. $D_x \cup \{r\}$ corresponds to each set exhibited in Cases 2-6 of the theorem proof (*n.b.*, Case 1 is for the beauty of the art, but useless)



1 Leader Election

($O(mn)$ messages, $O(m)$ rounds, and $O(\delta_u + B)$ bits for messages and nodes, where B the number of bits to store an identifier)

2 BFS Spanning Tree Construction (with initialization and termination detection at the leader)

($O(D)$ rounds, $O(n.m)$ messages of $O(\log D)$ bits, and $O(\log D + \log \Delta)$ bits per node)

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3 Propagation of Information with Feedback in the tree:

Broadcast $\Rightarrow$ Node $(k + 1)$ -Coloring,

Feedback $\Rightarrow$ Computation of the D -set sizes

($O(D)$ rounds, $O(n)$ messages of $O(k \cdot \log n)$ bits, and $O(\log \Delta + k \log n)$ bits per node)

Distributed Computation

1 Leader Election

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4 Propagation of Information with Feedback in the tree:

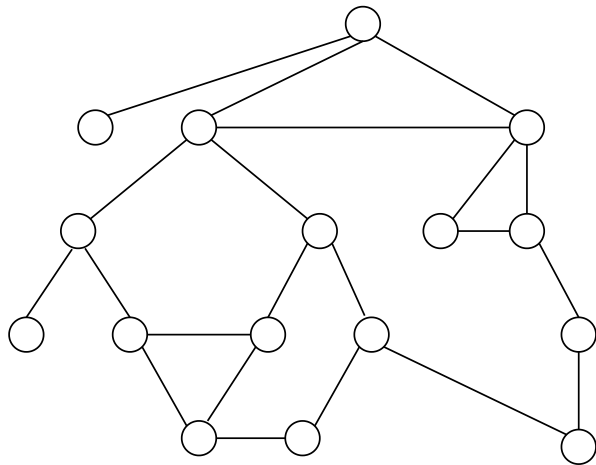
Broadcast $\Rightarrow$ clusterhead assignment and cluster coloring,

Feedback $\Rightarrow$ Termination Detection at the Leader

($O(D)$ rounds, $O(n)$ messages of $O(\log k + \log B)$ bits, and $O(\log B)$ bits per node)

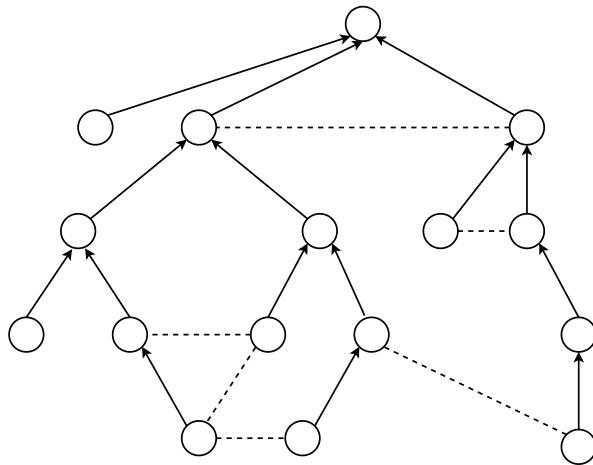
5 Routing Set-Up

Example with $k = 2$



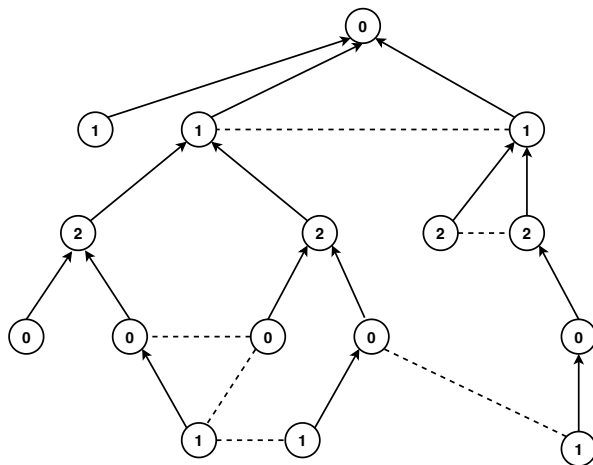
Network

Example with $k = 2$



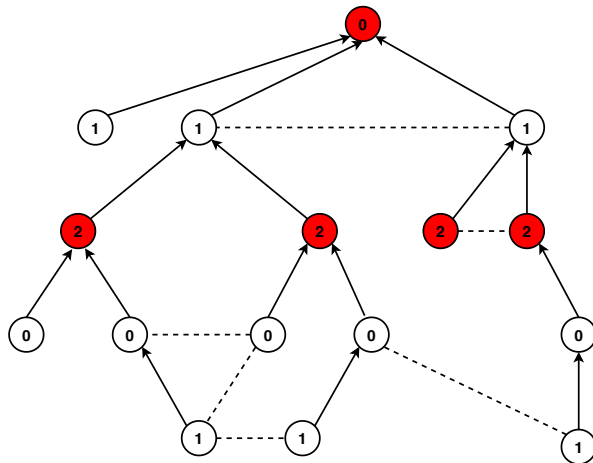
(BFS) spanning tree

Example with $k = 2$



Node $(k + 1)$ -Coloring

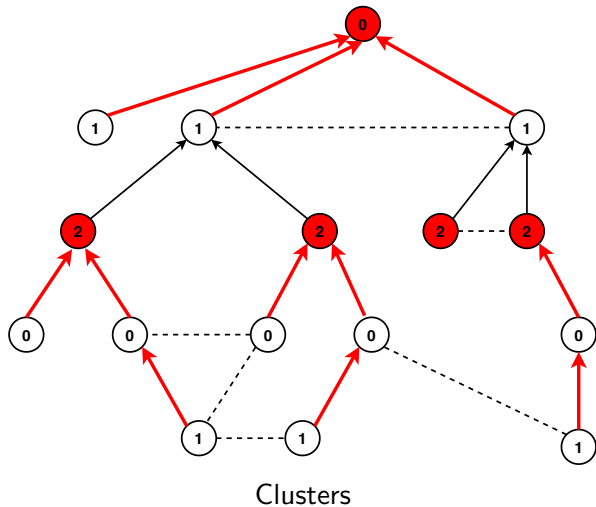
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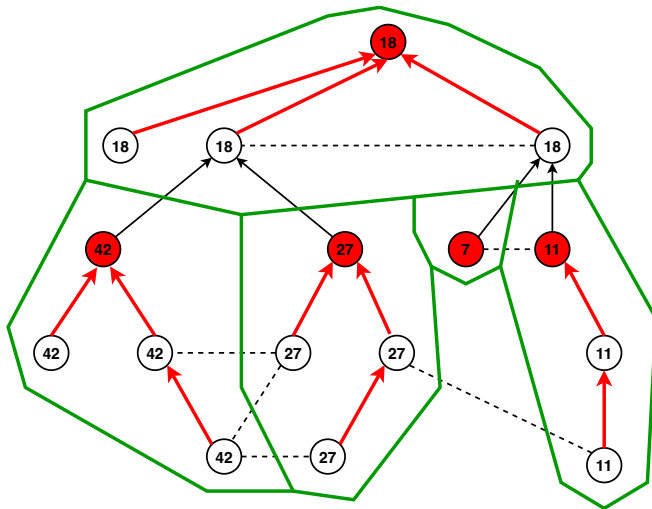
k -dominating set

$$|D_0| = 6, |D_1| = 6, |D_2| = 4$$

Example with $k = 2$

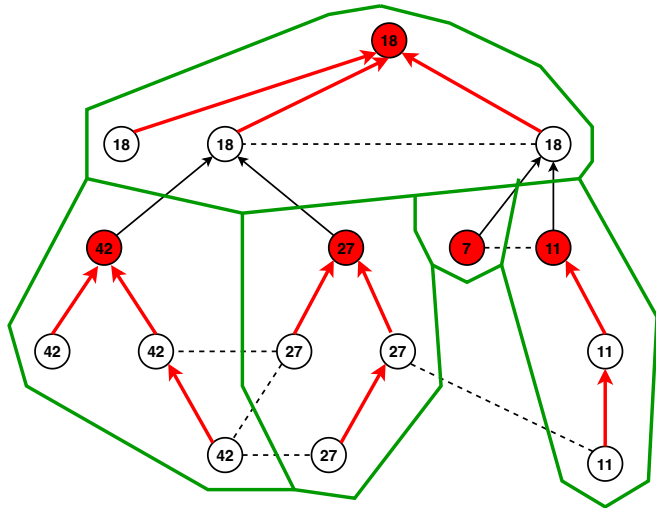


Example with $k = 2$



Clustering

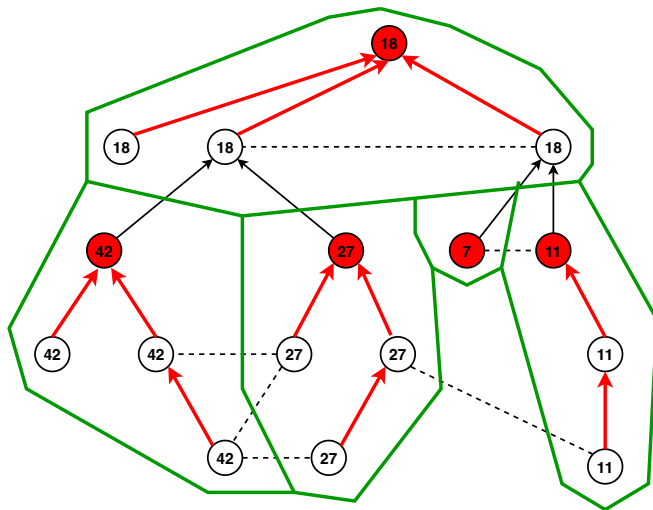
Example with $k = 2$



Clustering

$$\text{\#clusters} = 5 \leq \lceil \frac{16}{2} \rceil = 6$$

Example with $k = 2$



Clustering

Remark: the spanning tree can be used for inter-cluster routing

Pros:

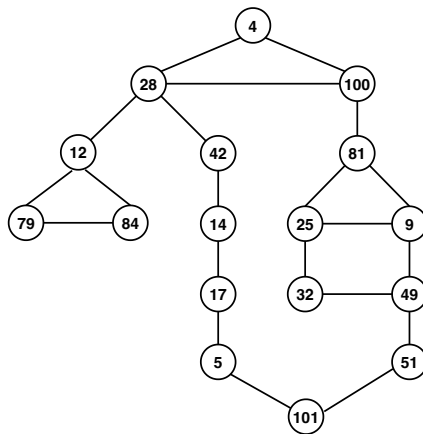
- Time-efficient computation
- BFS $\Rightarrow$ short paths

Cons:

- Memory requirement and message size: $\Omega(k \log n)$

Exercise

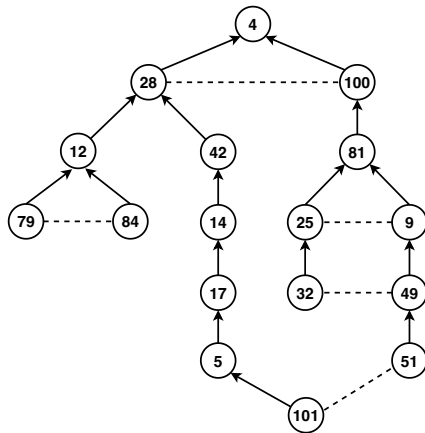
Compute a clustering using the algorithm of [3] with $k = 2$



A network

Exercise

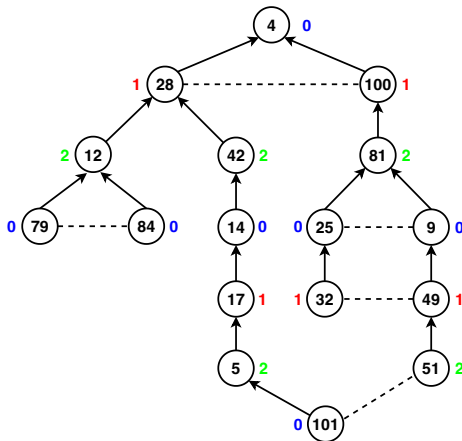
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(BFS) spanning tree

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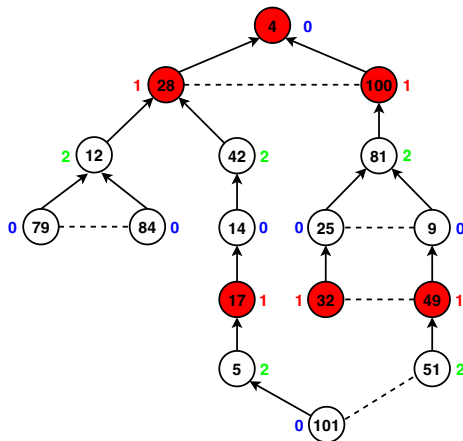
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Coloring

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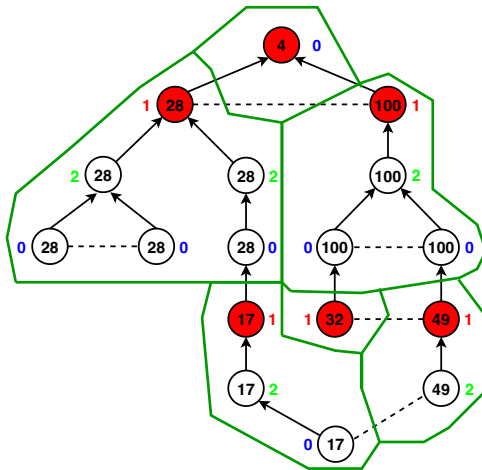
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Color choice: $|D_0| = 7$, $|D_1| = 5$, $|D_2| = 5$

Exercise

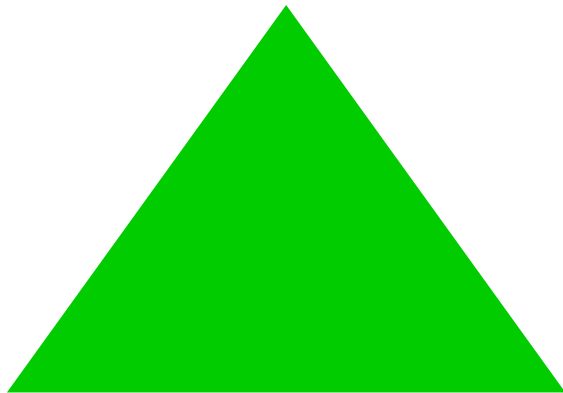
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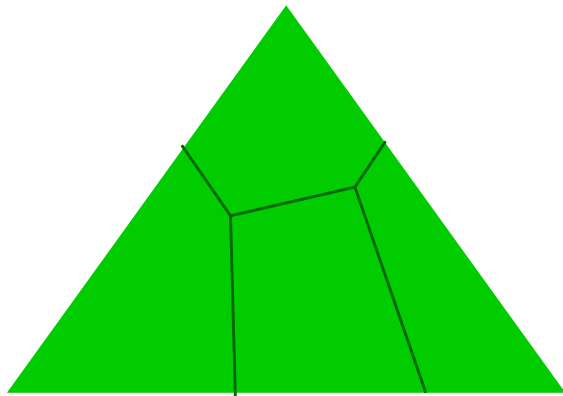
Clustering, **#clusters** = $6 \leq \lceil \frac{17}{3} \rceil = 6$

Roadmap

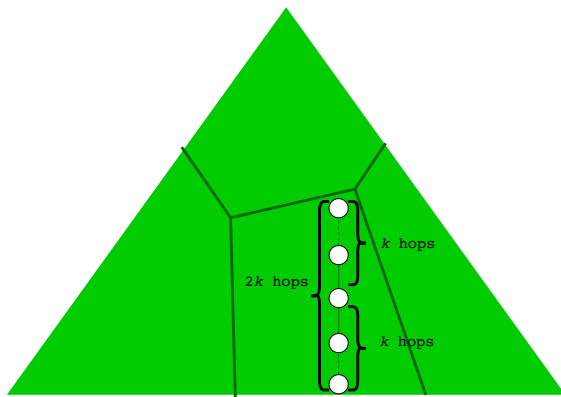
- 1 Introduction
- 2 Examples of clustering
 - Clustering of [3]
 - Clustering of [1]
- 3 Routing in a Clustering
- 4 References



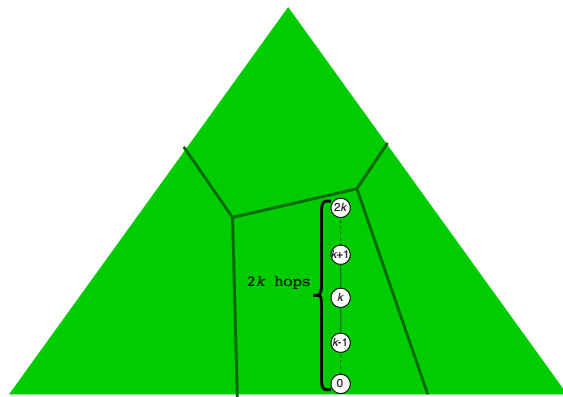
A (spanning) in-tree T



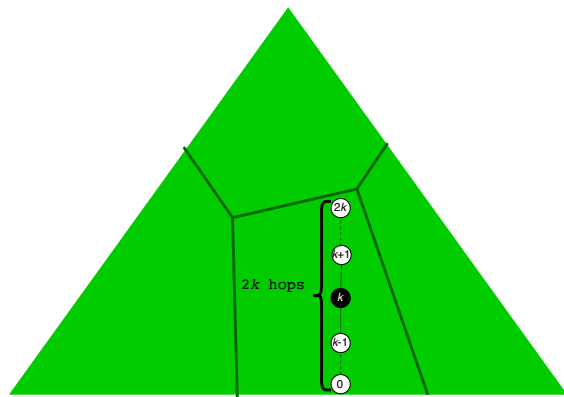
A (spanning) in-tree T with clusters



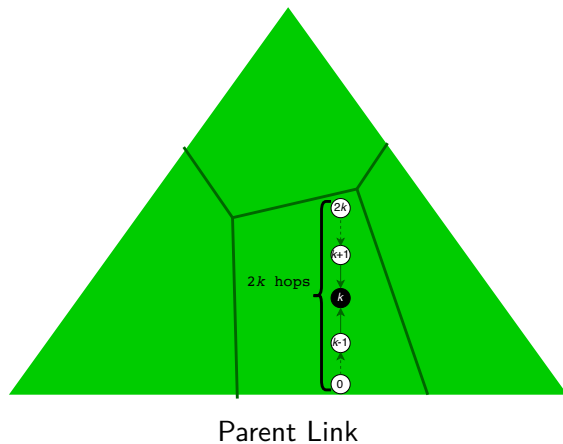
Build paths of $2k$ hops ($2k + 1$ nodes)

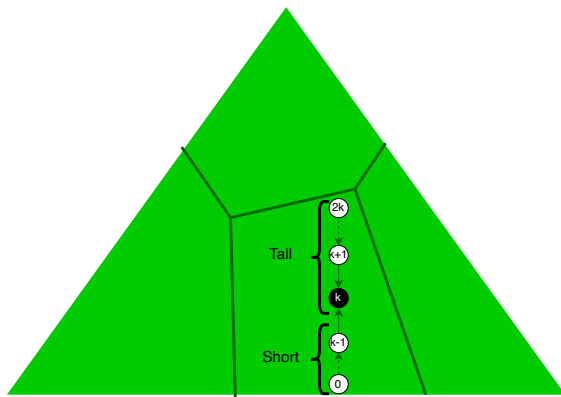


Node numbering

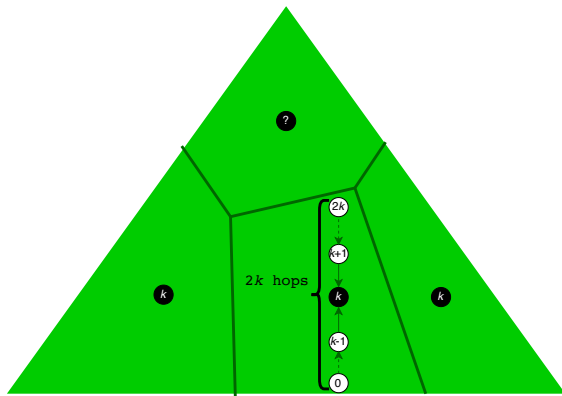


Node labeled k : clusterhead

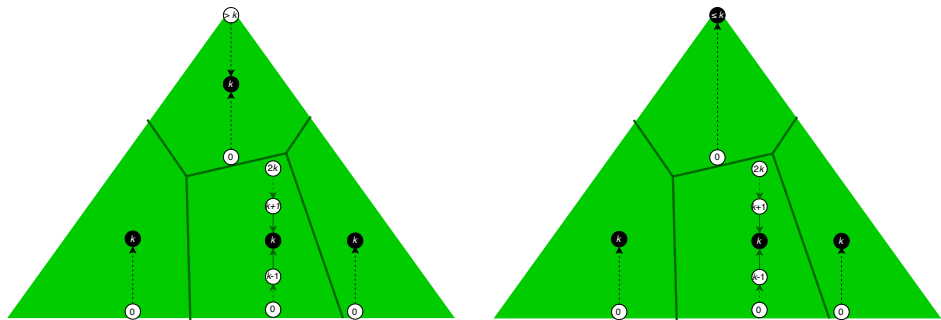




Tall and short nodes

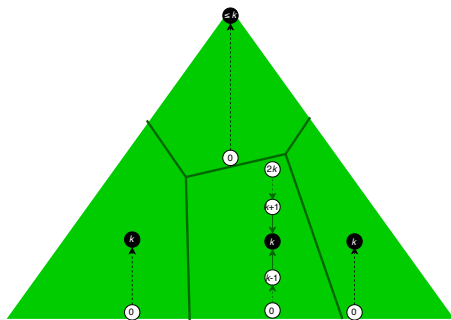
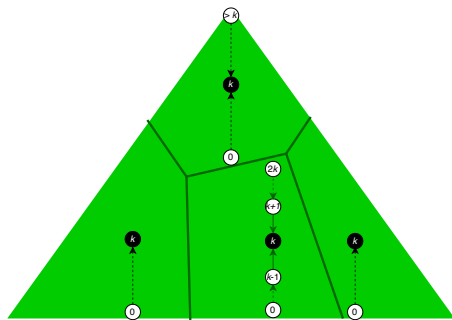


Issue: cluster of the root



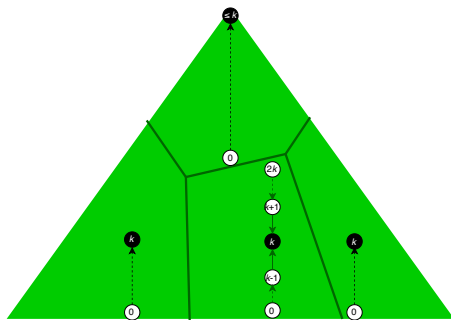
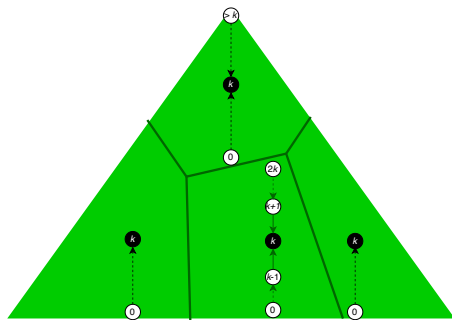
2 cases

Number of clusterheads



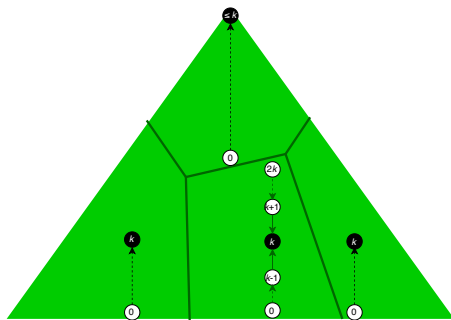
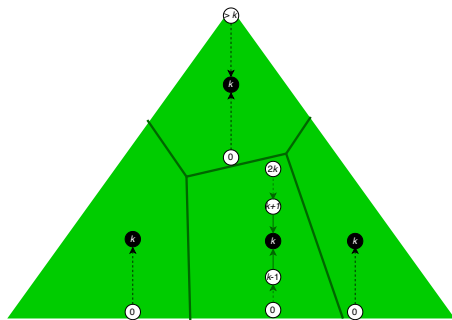
- 1 Non-root cluster: at least $k + 1$ nodes

Number of clusterheads



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Number of clusterheads



- 1 Non-root cluster: at least $k + 1$ nodes
- 2 1 cluster with the root + at most $\lfloor \frac{n-1}{k+1} \rfloor$ non-root cluster
- 3 $\# \text{Clusters} \leq 1 + \lfloor \frac{n-1}{k+1} \rfloor = \lfloor \frac{n+k}{k+1} \rfloor \leq \lceil \frac{n}{k+1} \rceil$

Node Numbering: function α

For every node p , $\alpha(p) \in [0..2k]$

- $\text{maxShort}(p) = \max(\{\alpha(q) \mid q \in \text{Children}(p) \wedge \alpha(q) < k\} \cup \{-1\})$
- $\text{minTall}(p) = \min(\{\alpha(q) \mid q \in \text{Children}(p) \wedge \alpha(q) \geq k\} \cup \{2k + 1\})$

if $\text{maxShort}(p) + \text{minTall}(p) \leq 2k - 2$ **then**

$$\alpha(p) = \text{minTall}(p) + 1$$

else

$$\alpha(p) = \text{maxShort}(p) + 1$$

end if

Tree Structure of clusters

$\alpha(p) = k$ or (p is the root and $\alpha(p) \leq k$): clusterhead (root of the cluster)

If p is not a clusterhead:

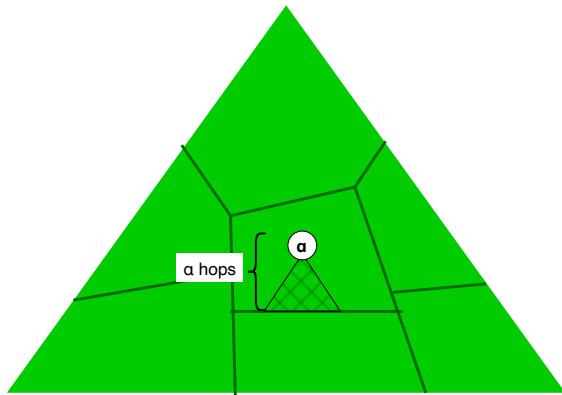
$\alpha(p) < k$: parent in the cluster := parent in the tree

$\alpha(p) > k$: parent in the cluster := a child q with $\alpha(q) = \min Tall(p)$

(ties can be broken arbitrarily, e.g., using identifiers)

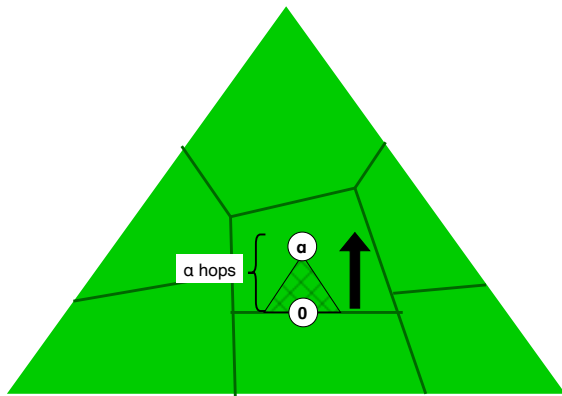
Meaning of α

$\alpha(p)$ is the distance from p to q where q is its furthest process in $T(p)$ that is in the same cluster as p



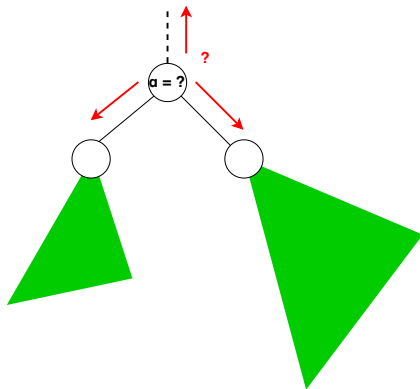
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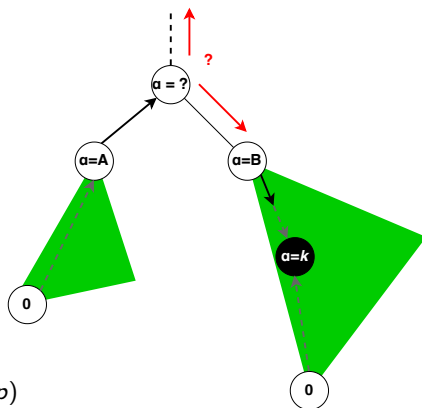


Bottom-up Computation

Computation of α

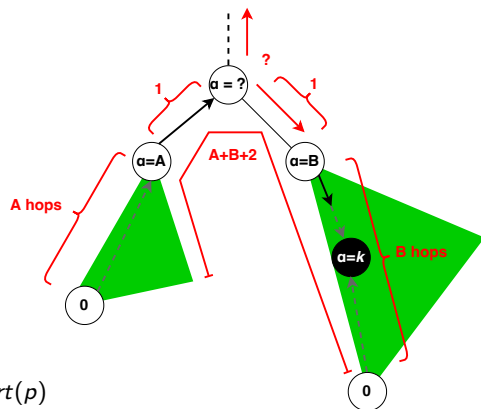


Computation of α



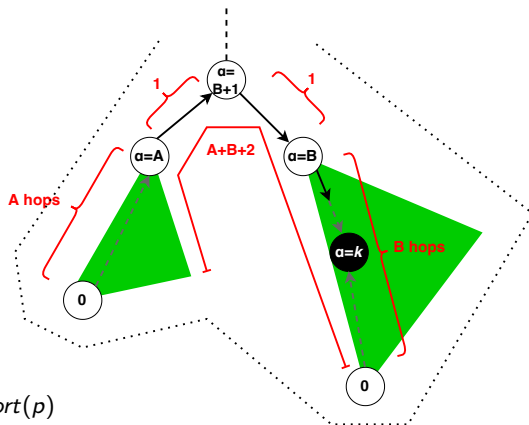
- $A = \maxShort(p)$
- $B = \minTall(p)$
- $0 \leq A < k \leq B \leq 2k$

Computation of α



- $A = \maxShort(p)$
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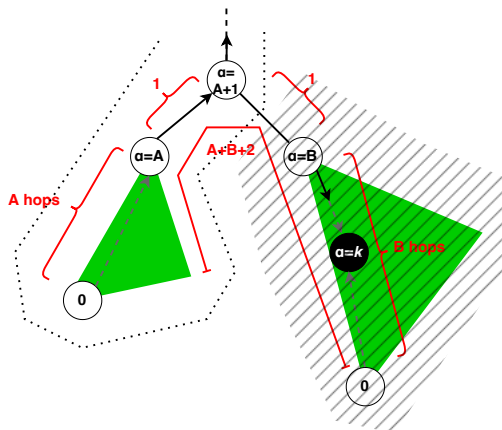
Computation of α



- $A = \maxShort(p)$
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- $0 \leq A < k \leq B \leq 2k$

Case $A + B + 2 \leq 2k$

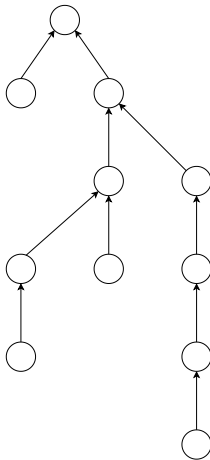
Computation of α



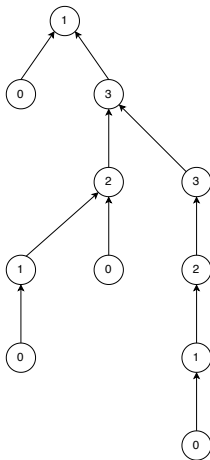
- $A = \maxShort(p)$
- $B = \minTall(p)$
- $0 \leq A < k \leq B \leq 2k$

Case $A + B + 2 > 2k$

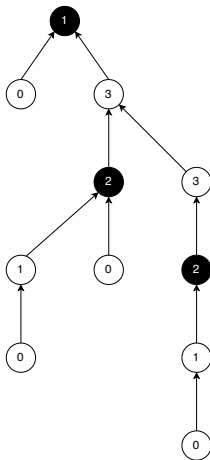
Example with $k = 2$



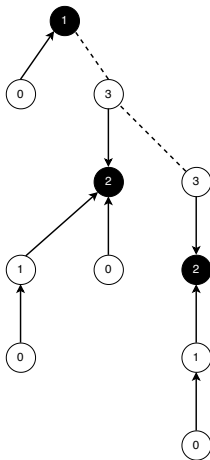
Example with $k = 2$



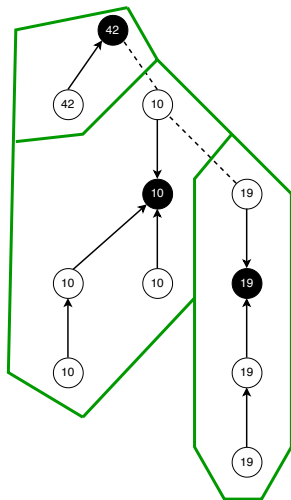
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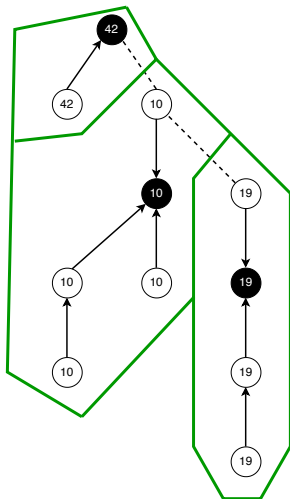
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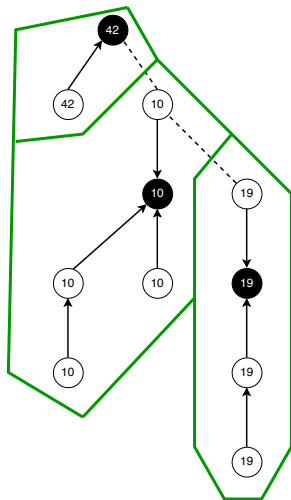


Example with $k = 2$



$$\text{\#clusters} = 3 \leq \lceil \frac{10}{3} \rceil = 4$$

Example with $k = 2$



Remark: the spanning tree can be used for inter-cluster routing

- Optimal in trees [1]
- Arbitrary Connected networks: at most $\lceil \frac{n}{k+1} \rceil$ clusters
- Unit Disk Graph (UDG): $7.2552k + O(1)$ -approximation of the optimal
- Quasi Unit Disk Graph (QUDG): $7.2552\lambda^2k + O(\lambda)$ -approximation of the optimal

UDG and QUDG are models for Wireless Sensor Networks Topologies

Distributed Computation

In Arbitrary Connected Networks

1 Leader Election

($O(mn)$ messages, $O(m)$ rounds, and $O(\delta_u + B)$ bits for messages and nodes, where B the number of bits to store an identifier)

2 BFS Spanning Tree Construction (with initialization and termination detection at the leader)

($O(D)$ rounds, $O(n.m)$ messages of $O(\log D)$ bits, and $O(\log D + \log \Delta)$ bits per node)

Distributed Computation

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Feedback $\Rightarrow$ Bottom-up computation of α + Parent pointer assignment

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Distributed Computation

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4 Propagation of Information with Feedback in the tree:

Broadcast + Feedback $\Rightarrow$ cluster coloring,

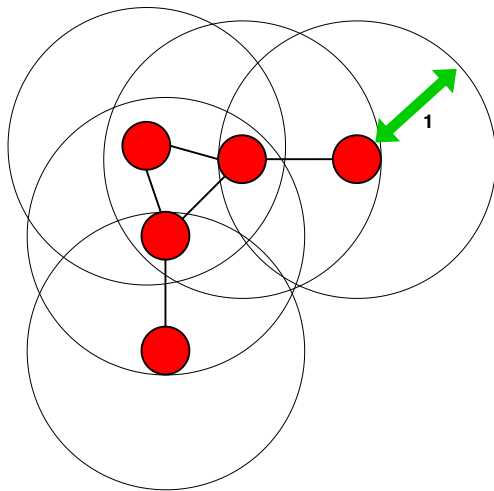
Feedback $\Rightarrow$ Termination Detection at the Leader

($O(D)$ rounds, $O(n)$ messages of $O(\log k + \log B)$ bits, and $O(\log B)$ bits per node)

5 Routing Set-Up

UDG: Unit Disk Graphs

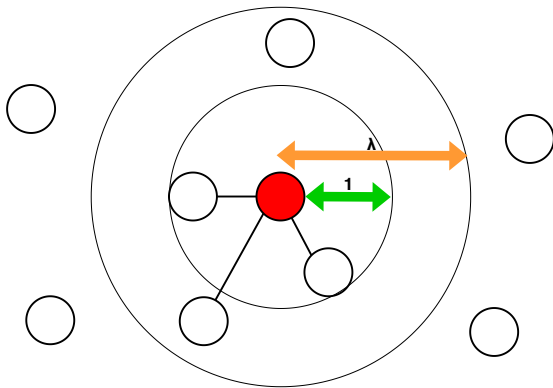
Nodes u and v are neighbors $\equiv \|u, v\| \leq 1$



QUDG: Quasi Unit Disk Graphs

Let $\lambda \geq 1$

- $\|u, v\| \leq 1 \Rightarrow u$ and v are neighbors
- u and v are neighbors $\Rightarrow \|u, v\| \leq \lambda$



Approximation of the optimal in UDGs

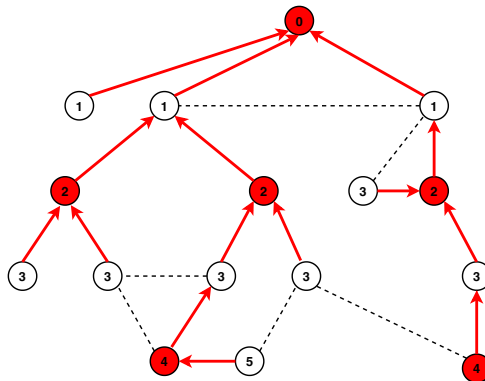
(at most M times the size of the optimal one)

Computing α on a MIS Tree of $G = (V, E)$

A spanning tree of G whose nodes at even level form a **maximal independent set**

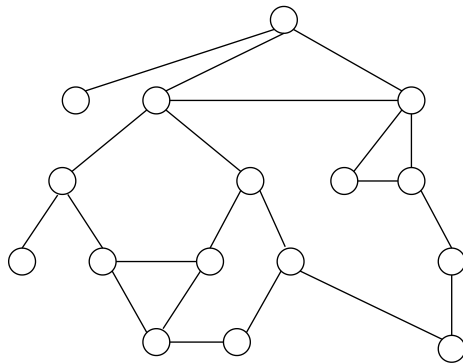
$S \subseteq V$ is a **maximal independent set** if

- S is independent: no two distinct nodes of S are neighbors in G
- S is maximal (by inclusion): no proper superset of S is independent



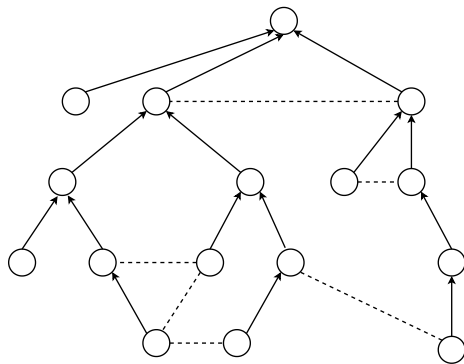
Approximation of the optimal in UDGs

Compute a MIS Tree



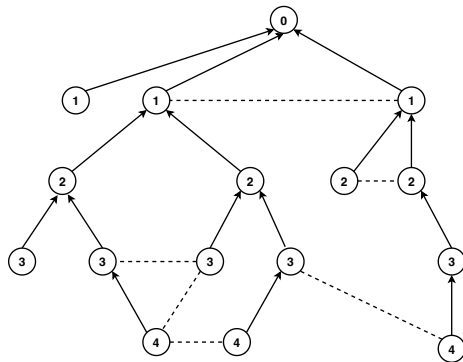
A network

Approximation of the optimal in UDGs



Approximation of the optimal in UDGs

Compute a MIS Tree



BFS Spanning Tree with levels

Approximation of the optimal in UDGs

Compute a MIS Tree

- $p.status \in \{In, Out\}$, initially *Out*
- DFS traversal of the spanning tree

At the 1st visit of node p :

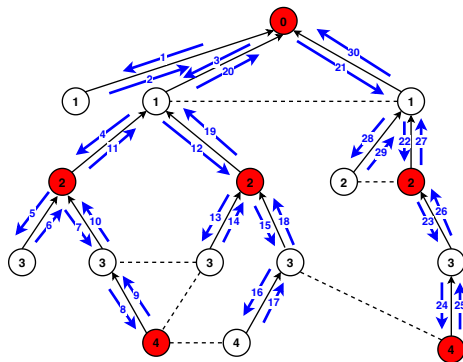
if every neighbor q of p satisfies
 $q.status = Out$ **then**

$p.status \leftarrow In$

else

$p.status \leftarrow Out$

end if



BFS Spanning Tree with MIS

Approximation of the optimal in UDGs

Compute a MIS Tree

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At the 1st visit of node p :

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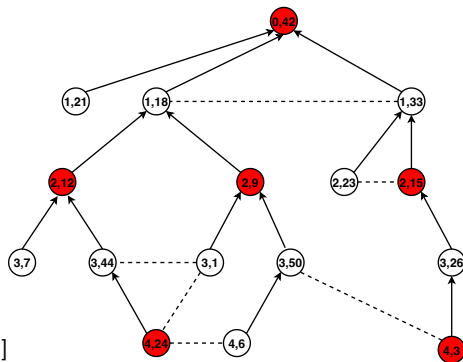
$p.status \leftarrow In$

else

$p.status \leftarrow Out$

end if

- **Key:** pair level/identifier
- **Total order on keys:**
 $(a, b) \prec (c, d) \equiv [a < c \vee (a = c \wedge b < d)]$



BFS Spanning Tree and MIS with keys

Approximation of the optimal in UDGs

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- DFS traversal of the spanning tree

At the 1st visit of node p :

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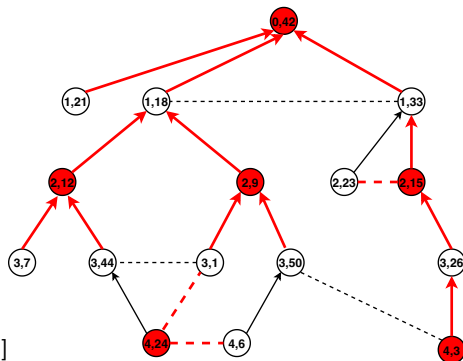
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- **Parent of non-root nodes:** the neighbor with a different status of smallest key



MIS tree

Approximation of the optimal in UDGs

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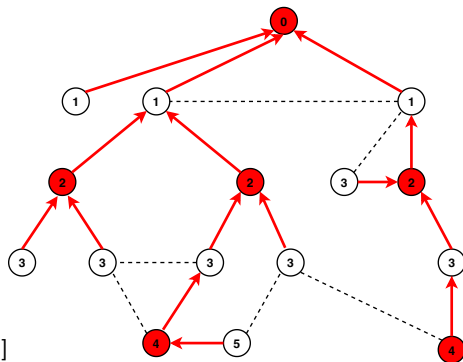
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MIS tree with levels

Approximation of the optimal in UDGs

Distributed computation of a MIS Tree

1 Leader Election

($O(mn)$ messages, $O(m)$ rounds, and $O(\delta_u + B)$ bits for messages and nodes, where B the number of bits to store an identifier)

2 BFS Spanning Tree Construction (with initialization and termination detection at the leader)

($O(D)$ rounds, $O(n.m)$ messages of $O(\log D)$ bits, and $O(\log D + \log \Delta)$ bits per node)

Approximation of the optimal in UDGs

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3 Token Circulation in the tree: first visit of a node $\Rightarrow$ collect status of neighbors (local message exchanges), then status assignment

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Approximation of the optimal in UDGs

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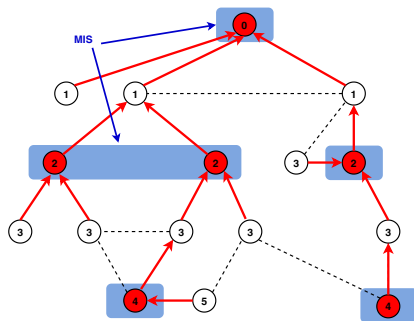
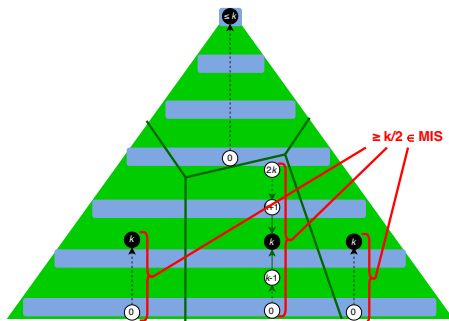
($O(D)$ rounds, $O(m)$ messages of $O(\log k + \log B)$ bits, and $O(\Delta(\log k + \log B))$ bits per node)

5 Clustering computation on the MIS tree

Approximation of the optimal in UDGs

k -clustering vs. MIS

Clr: partition into clusters

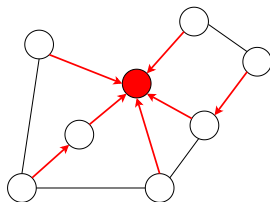


$$(|\text{Clr}| - 1) \cdot \frac{k}{2} \leq |\text{MIS}| - 1$$

Approximation of the optimal in UDGs

Independent Set vs. Optimal k -clustering Clr_{opt}

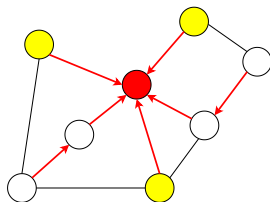
- Let C be any cluster of Clr_{opt}



Approximation of the optimal in UDGs

Independent Set vs. Optimal k -clustering Clr_{opt}

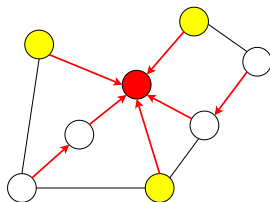
- Let C be any cluster of Clr_{opt}
- Let I be any independent set
- $I \cap C$



Approximation of the optimal in UDGs

Independent Set vs. Optimal k -clustering Clr_{opt}

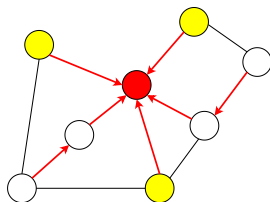
- Let C be any cluster of Clr_{opt}
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- **UDG:**
 $\forall p, q \in I, p \neq q \Rightarrow \|p, q\| > 1$



Approximation of the optimal in UDGs

Independent Set vs. Optimal Clr_{opt}

- Let C be any cluster of Clr_{opt}
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 $\forall p, q \in I, p \neq q \Rightarrow \|p, q\| > 1$



Theorem (Folkman & Graham, [4])

Let X be a compact convex region. Let $Y \subseteq X$ s.t. $\forall p, q \in Y, (p \neq q \Rightarrow \|p, q\| \geq 1)$.

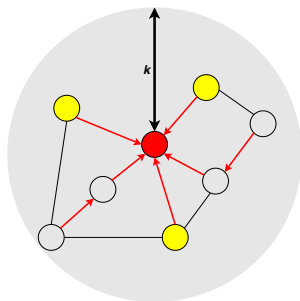
$$|Y| \leq \left\lfloor \frac{2 \cdot A(X)}{\sqrt{3}} + \frac{P(X)}{2} + 1 \right\rfloor$$

where $A(X)$ and $P(X)$ are respectively the area and perimeter of X .

Approximation of the optimal in UDGs

Independent Set vs. Optimal k -clustering Clr_{opt}

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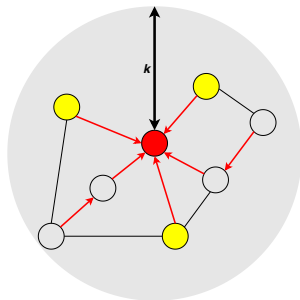
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Approximation of the optimal in UDGs

Independent Set vs. Optimal k -clustering Clr_{opt}

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- Let I be any independent set
- $I \cap C$
- **UDG:**
 $\forall p, q \in I, p \neq q \Rightarrow \|p, q\| > 1$



$$A(X) = \pi \cdot k^2, P(X) = 2\pi \cdot k$$

- $|I \cap C| \leq \lfloor \frac{2 \cdot \pi \cdot k^2}{\sqrt{3}} + \pi \cdot k + 1 \rfloor$
- $|I| \leq \lfloor \frac{2 \cdot \pi \cdot k^2}{\sqrt{3}} + \pi \cdot k + 1 \rfloor \cdot |Clr_{opt}|$

- $|MIS| \leq \lfloor \frac{2 \cdot \pi \cdot k^2}{\sqrt{3}} + \pi \cdot k + 1 \rfloor \cdot |Clr_{opt}|$

- $(|Clr| - 1) \cdot \frac{k}{2} \leq |MIS| - 1$

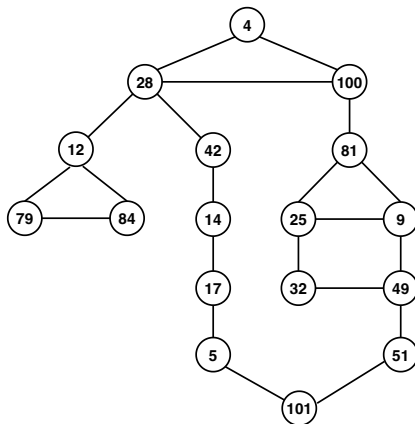
$$\Rightarrow |Clr| \leq 1 - \frac{2}{k} + \left(\frac{4\pi \cdot k}{\sqrt{3}} + 2\pi + \frac{2}{k} \right) \cdot |Clr_{opt}|$$

$\Rightarrow 7, 2552k + O(1)$ —competitive in UDG

Generalization to QUDG: $7, 2552\lambda^2 k + O(1)$ —competitive in QUDG

Exercise

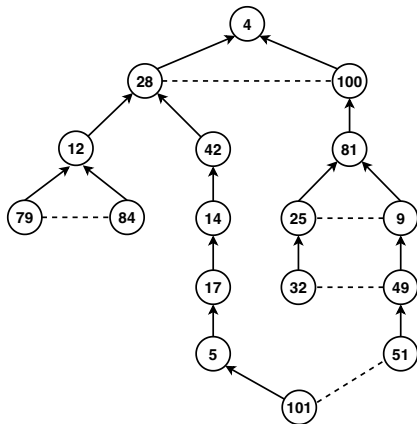
Compute a clustering using the algorithm of [1] **with a MIS tree** and $k = 2$



A network

Exercise

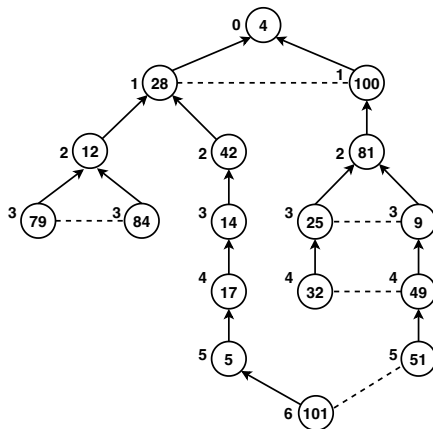
Compute a clustering using the algorithm of [1] **with a MIS tree** and $k = 2$



BFS spanning tree

Exercise

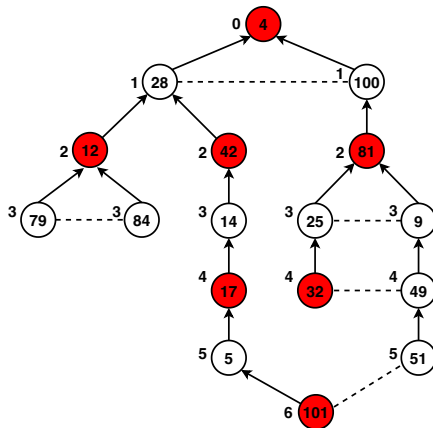
Compute a clustering using the algorithm of [1] **with a MIS tree** and $k = 2$



BFS spanning tree with levels

Exercise

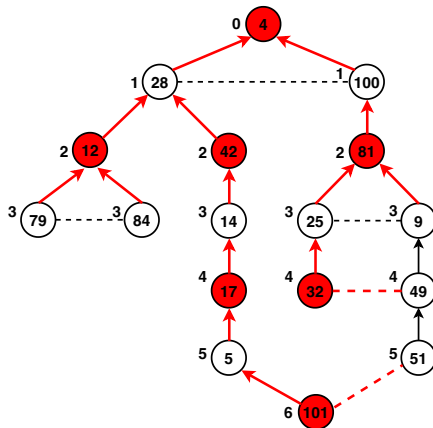
Compute a clustering using the algorithm of [1] **with a MIS tree** and $k = 2$



BFS spanning tree with levels + Independent Set

Exercise

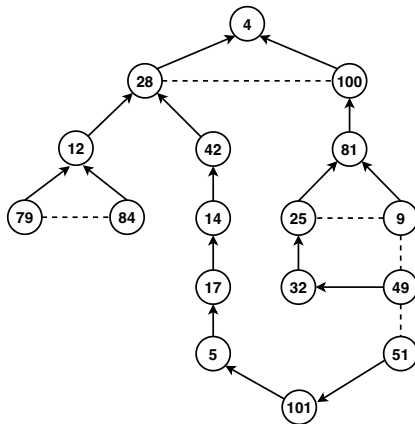
Compute a clustering using the algorithm of [1] **with a MIS tree** and $k = 2$



BFS spanning tree with levels + Independent Set + MIS tree

Exercise

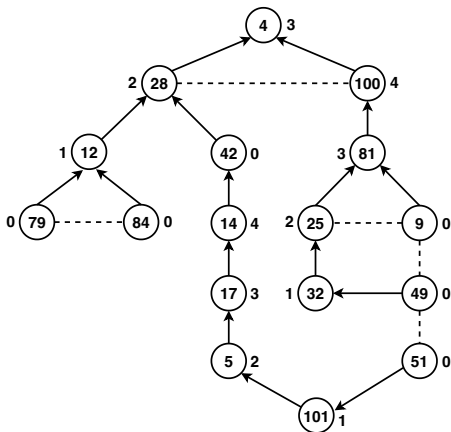
Compute a clustering using the algorithm of [1] **with a MIS tree** and $k = 2$



MIS tree

Exercise

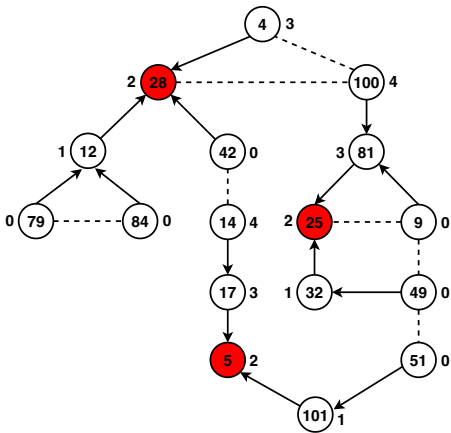
Compute a clustering using the algorithm of [1] **with a MIS tree** and $k = 2$



α -computation

Exercise

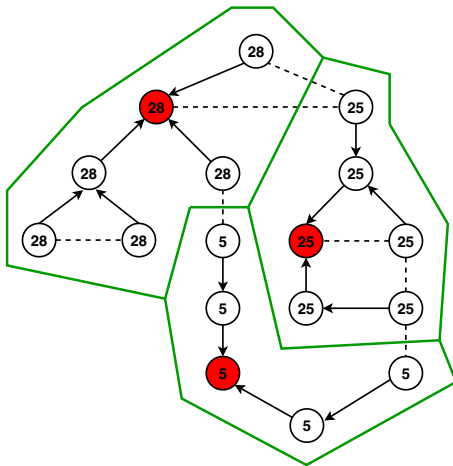
Compute a clustering using the algorithm of [1] **with a MIS tree** and $k = 2$



Spanning forest

Exercise

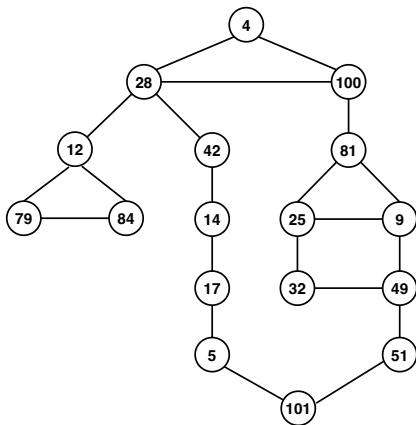
Compute a clustering using the algorithm of [1] **with a MIS tree** and $k = 2$



Clustering, **#clusters** = $3 \leq \lceil \frac{17}{3} \rceil = 6$

Exercise

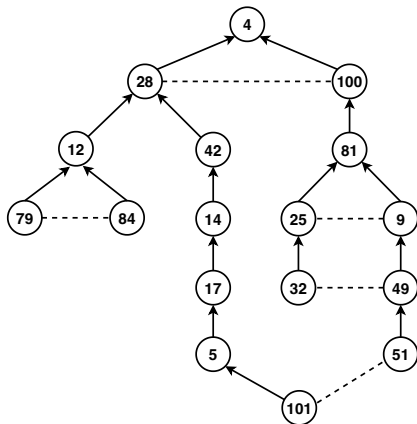
Compute a clustering using the algorithm of [1] **without a MIS tree** and with $k = 2$



A network

Exercise

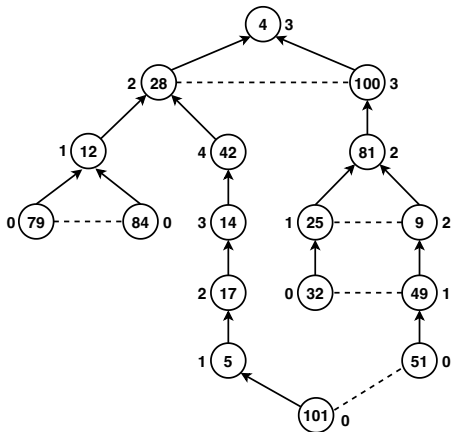
Compute a clustering using the algorithm of [1] **without a MIS tree** and with $k = 2$



(BFS) spanning tree

Exercise

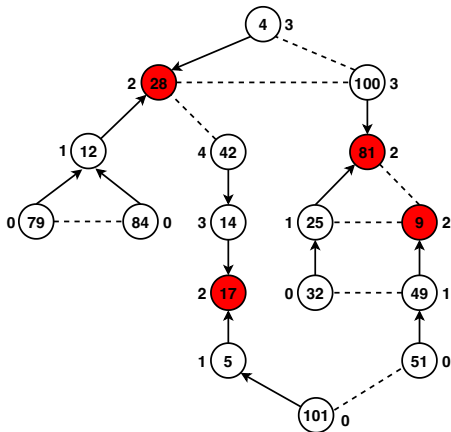
Compute a clustering using the algorithm of [1] **without a MIS tree** and with $k = 2$



α -computation

Exercise

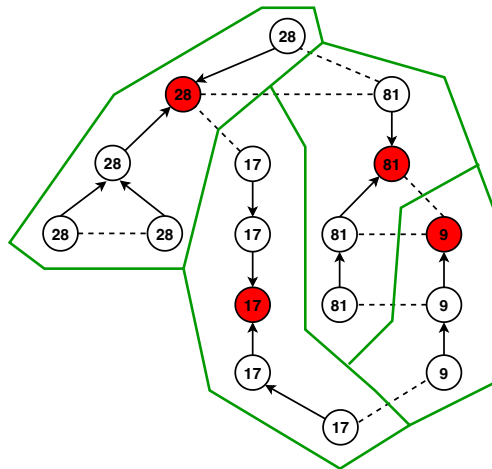
Compute a clustering using the algorithm of [1] **without a MIS tree** and with $k = 2$



Spanning forest

Exercise

Compute a clustering using the algorithm of [1] **without a MIS tree** and with $k = 2$



Clustering, **#clusters** = $4 \leq \lceil \frac{17}{3} \rceil = 6$

Roadmap

- 1 Introduction
- 2 Examples of clustering
 - Clustering of [3]
 - Clustering of [1]
- 3 Routing in a Clustering
- 4 References

Routing from p to q

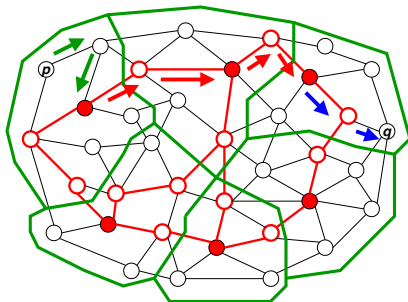
Address: (label of q 's clusterhead, label of q)

Phases: 3 phases distinguished using **3 colors**.

- 1 Route the packet from p to its clusterhead (intra-clustering)
- 2 Route the packet from the clusterhead of p to the clusterhead of q (inter-clustering)
- 3 Route the packet from the clusterhead of q to q (intra-clustering)

Different routing methods can be used for each phase, e.g.,

- 1 Parent links to route from p to its clusterhead
- 2 Shortest paths¹ to route from the clusterhead of p to the clusterhead of q
- 3 Interval routing from the clusterhead of q to q



¹in the network or in the tree computed during the clustering computation

Shortest Paths for inter-cluster routing

GLOBAL:

- 1 The leader launches a PIF to collect the identifiers of all clusterheads
- 2 The leader launches a PIF to broadcast the identifiers of all clusterheads
- 3 For each node p and each clusterhead q :
 - $Par_p[q] \leftarrow p$
 - if $p \neq q$, $D_p[q] \leftarrow \infty$, else $D_p[q] = 0$
- 4 **Bellman-Ford:** for each node p , each clusterhead q , and every neighbor v of p
 - if $D_p[q] > D_v[q] + 1$, then $D_p[q] \leftarrow D_v[q] + 1$; $Par_p[q] \leftarrow v$

LOCAL:

- 1 Each node p creates cells $Par_p[q]$ and $D_p[q]$ for its clusterhead q and informs its neighbors of the existence of q
- 2 $Par_p[q] \leftarrow p$ and if $p \neq q$, $D_p[q] \leftarrow \infty$, else $D_p[q] = 0$
- 3 If p learns the existence of some clusterhead r , p creates cells $Par_p[r]$ and $D_p[r]$, initialized to p and ∞ resp.

See [6] for further details.

Using the Spanning Tree for inter-cluster routing

Thank to the cluster colors, local detection of tree edge $\{p, q\}$ between two different clusters

- 1 Add $\{p, q\}$ to the subgraph connecting clusterheads
- 2 Add the cluster path from p to its clusterhead to the subgraph connecting clusterheads
- 3 Add the cluster path from q to its clusterhead to the subgraph connecting clusterheads

We obtain a tree

Using the Spanning Tree for inter-cluster routing

Thank to the cluster colors, local detection of tree edge $\{p, q\}$ between two different clusters

- ➊ Add $\{p, q\}$ to the subgraph connecting clusterheads
- ➋ Add the cluster path from p to its clusterhead to the subgraph connecting clusterheads
- ➌ Add the cluster path from q to its clusterhead to the subgraph connecting clusterheads

We obtain a tree

Apply the tree labeling scheme or an PLS on the tree to allow routing between clusters

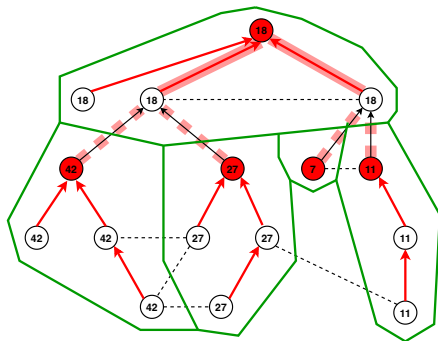
Using the Spanning Tree for inter-cluster routing

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Apply the tree labeling scheme or an PLS on the tree to allow routing between clusters



Clustering of [3]

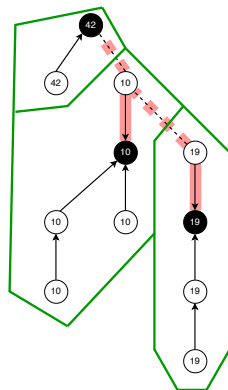
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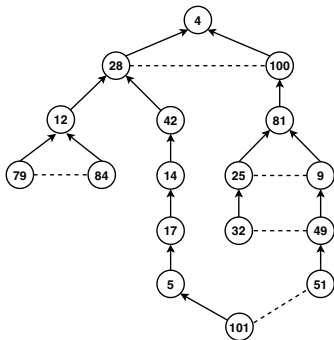
We obtain a tree

Apply the tree labeling scheme or an PLS on the tree to allow routing between clusters

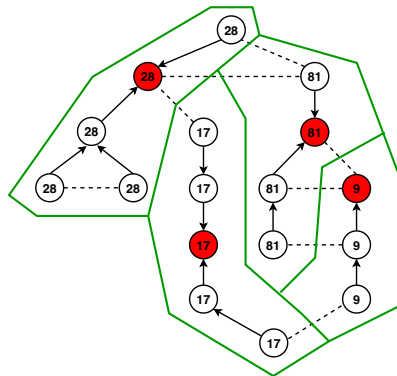


Clustering of [1]

Exercise: subgraph connecting clusterheads?

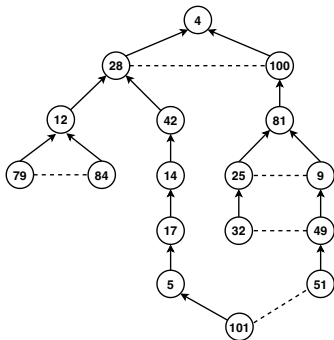


Spanning tree

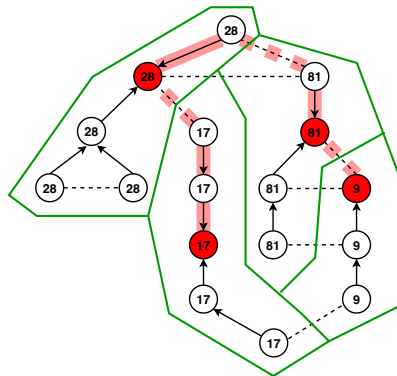


Clustering

Exercise: subgraph connecting clusterheads



Spanning tree



Clustering

Subgraph connecting clusterheads

Optimal subgraph for the Inter-cluster Routing

Steiner Tree

Non-negative edge weights.

Subset of nodes: set of terminals

Compute a tree of minimum weight that

- contains all terminals (but may include additional vertices) and
- minimizes the total weight of its edges.

Here, terminals are clusterheads

However, $\mathcal{NP}$ -hard [5]

But approximation algorithms exist!

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References

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- [3] A. K. Datta, L. L. Larmore, S. Devismes, K. Heurtefeux, and Y. Rivierre.
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- [5] M. R. Garey and D. S. Johnson.
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- [6] G. Tel.
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