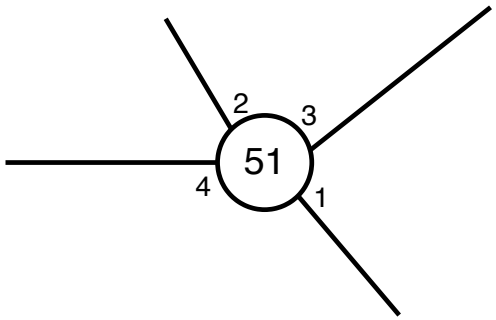
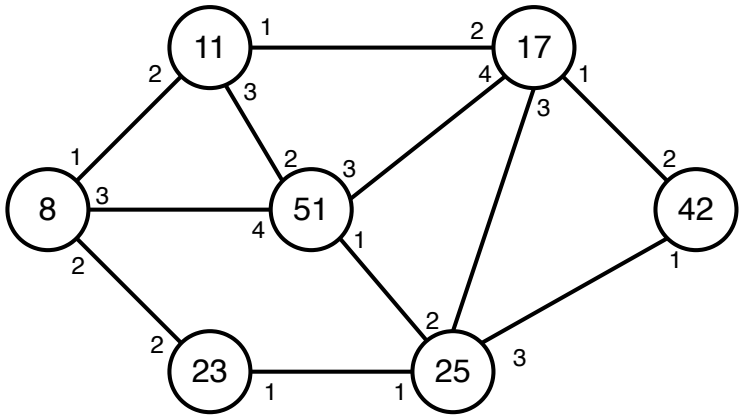
How to route a packet from 51 to 42?

Using Local Information Only . . .

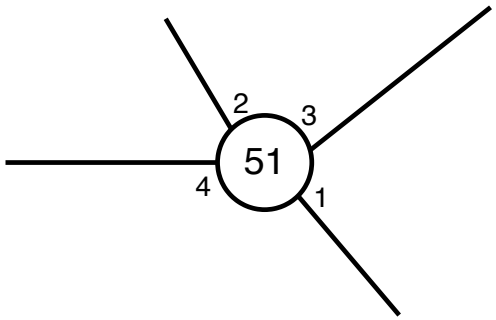


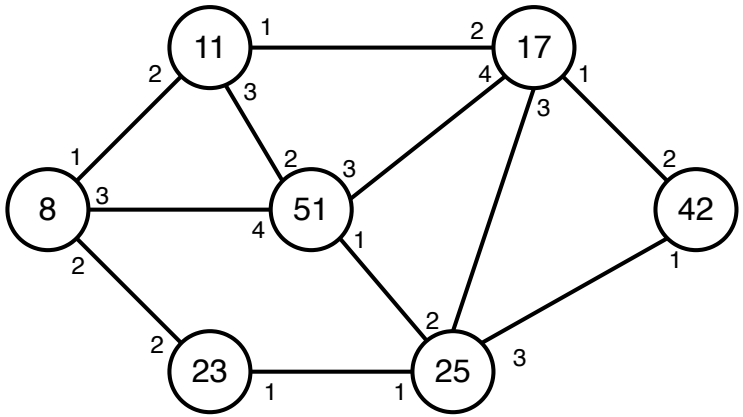


- The process identifier
- Port numbers of incident channels

How to route a packet from 51 to 42?

Using Local Information Only . . .



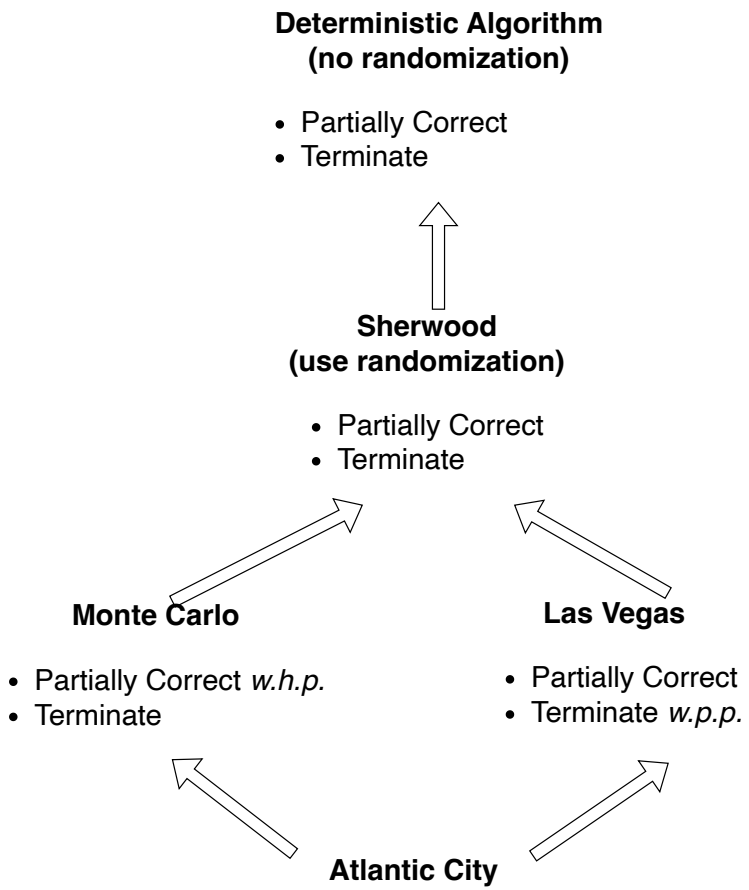


- The process identifier
- Port numbers of incident channels

In an arbitrary connected bidirectional network, without any further information:

only **randomization** can help!

Random Algorithms



w.p.p. = with (strictly) positive probability
w.h.p. = with high probability, *i.e.*, the probability depends on a parameter x such that the probability converges to 1 when x goes to the infinite (*w.h.p.* $\Rightarrow$ *w.p.p.*)

Remark: the Quicksort algorithm where the pivot is randomly chosen is a Sherwood algorithm.

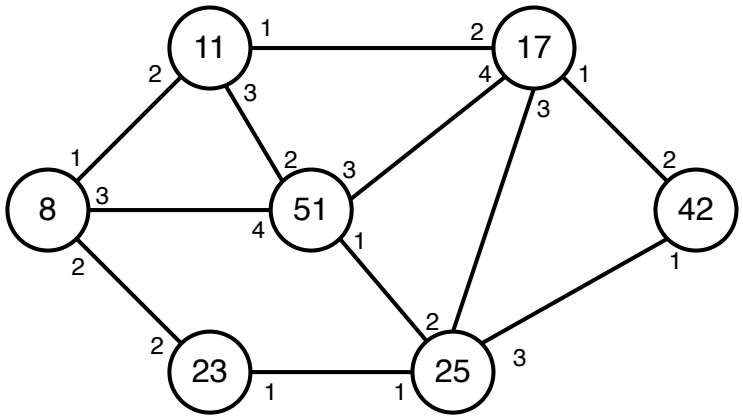
Random Local Algorithm (Las Vegas Algorithm)

Given a packet p with destination label d at node u.

if $d = u$ **then**
    `deliver` p
else
    pick $i \in \{1, \ldots, \delta_u\}$ according to P_u
    send p via port number i
end if

P_u is a probability distribution:

- $\forall i \in \{1, \ldots, \delta_u\}$, $P_u(i)$ gives the probability of picking i
- $P_u : \{1, \ldots, \delta_u\} \to [0, 1]$ such that $\sum_{i \in \{1, \ldots, \delta_u\}} P_u(i) = 1$



Random Local Algorithm (Las Vegas Algorithm)

Given a packet p with destination label d at node u.

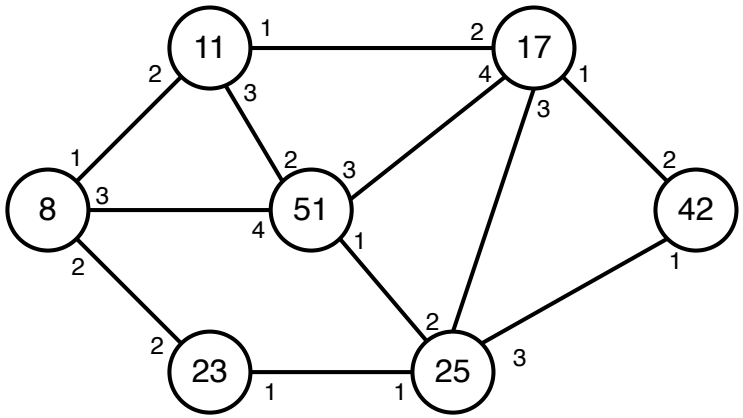
if $d = u$ **then**
    `deliver` p
else
    pick $i \in \{1, \ldots, \delta_u\}$ according to P_u
    send p via port number i
end if

P_u is a probability distribution:

- $\forall i \in \{1, \ldots, \delta_u\}$, $P_u(i)$ gives the probability of picking i
- $P_u : \{1, \ldots, \delta_u\} \to [0, 1]$ such that $\sum_{i \in \{1, \ldots, \delta_u\}} P_u(i) = 1$
- For example, P_u may be a uniform distribution:

$$\forall i \in \{1, \ldots, \delta_u\}, P_u(i) = \frac{1}{\delta_u}$$

E.g., $P_8(1) = P_8(2) = P_8(3) = \frac{1}{3}$



Random Local Algorithm (Las Vegas Algorithm)

Given a packet p with destination label d at node u.

if $d = u$ **then**
  `deliver` p
else
  pick $i \in \{1, \ldots, \delta_u\}$ according to P_u
  send p via port number i
end if

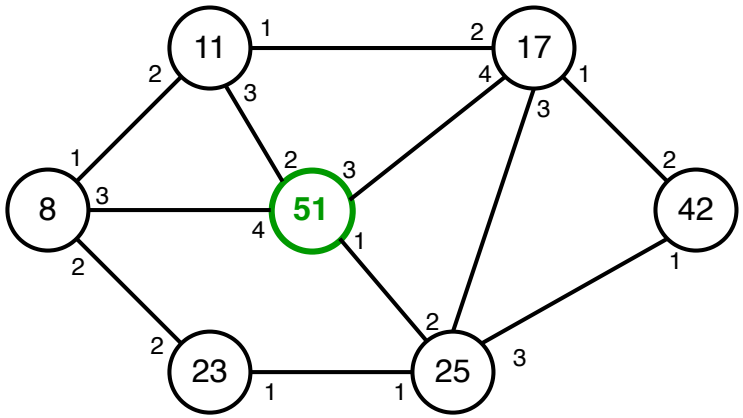
P_u is a probability distribution:

- $\forall i \in \{1, \ldots, \delta_u\}$, $P_u(i)$ gives the probability of picking i
- $P_u : \{1, \ldots, \delta_u\} \to [0, 1]$ such that $\sum_{i \in \{1, \ldots, \delta_u\}} P_u(i) = 1$
- For example, P_u may be a uniform distribution:

$$\forall i \in \{1, \ldots, \delta_u\}, P_u(i) = \frac{1}{\delta_u}$$

E.g., $P_8(1) = P_8(2) = P_8(3) = \frac{1}{3}$

Routing from 51 to 42
(*e.g.*, using uniform distribution)



Routing path: 51

Random Local Algorithm (Las Vegas Algorithm)

Given a packet p with destination label d at node u.

if $d = u$ **then**
  `deliver` p
else
  pick $i \in \{1, \ldots, \delta_u\}$ according to P_u
  send p via port number i
end if

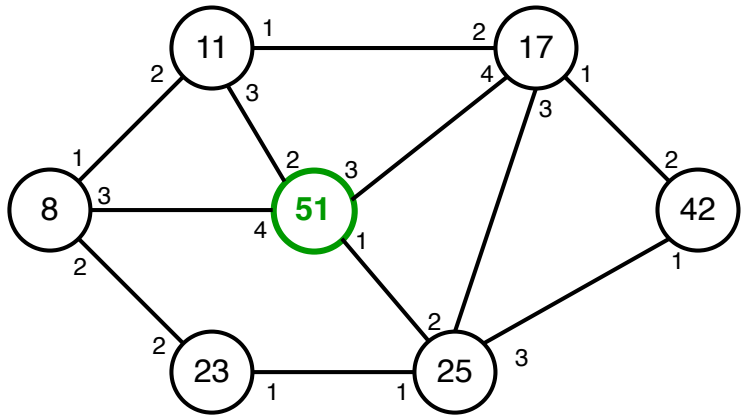
P_u is a probability distribution:

- $\forall i \in \{1, \ldots, \delta_u\}$, $P_u(i)$ gives the probability of picking i
- $P_u : \{1, \ldots, \delta_u\} \to [0, 1]$ such that $\sum_{i \in \{1, \ldots, \delta_u\}} P_u(i) = 1$
- For example, P_u may be a uniform distribution:

$$\forall i \in \{1, \ldots, \delta_u\}, P_u(i) = \frac{1}{\delta_u}$$

E.g., $P_8(1) = P_8(2) = P_8(3) = \frac{1}{3}$

Routing from 51 to 42
(*e.g.*, using uniform distribution)



Routing path: 51
Pick 3

Random Local Algorithm (Las Vegas Algorithm)

Given a packet p with destination label d at node u.

if $d = u$ **then**
  `deliver` p
else
  pick $i \in \{1, \ldots, \delta_u\}$ according to P_u
  send p via port number i
end if

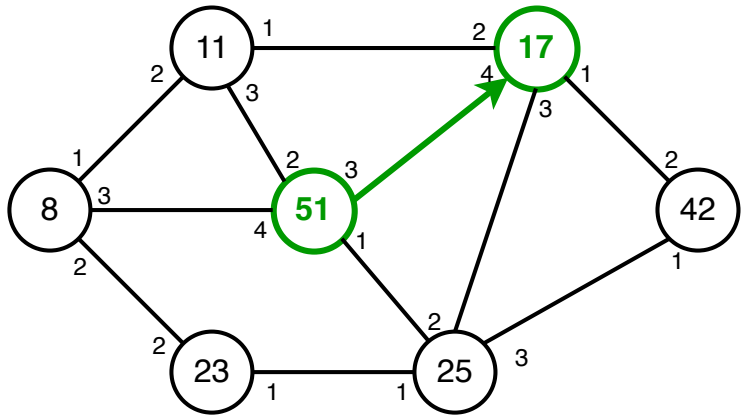
P_u is a probability distribution:

- $\forall i \in \{1, \ldots, \delta_u\}$, $P_u(i)$ gives the probability of picking i
- $P_u : \{1, \ldots, \delta_u\} \to [0, 1]$ such that $\sum_{i \in \{1, \ldots, \delta_u\}} P_u(i) = 1$
- For example, P_u may be a uniform distribution:

$$\forall i \in \{1, \ldots, \delta_u\}, P_u(i) = \frac{1}{\delta_u}$$

E.g., $P_8(1) = P_8(2) = P_8(3) = \frac{1}{3}$

Routing from 51 to 42
(*e.g.*, using uniform distribution)



Routing path: 51,17

Random Local Algorithm (Las Vegas Algorithm)

Given a packet p with destination label d at node u.

if $d = u$ **then**
  `deliver` p
else
  pick $i \in \{1, \ldots, \delta_u\}$ according to P_u
  send p via port number i
end if

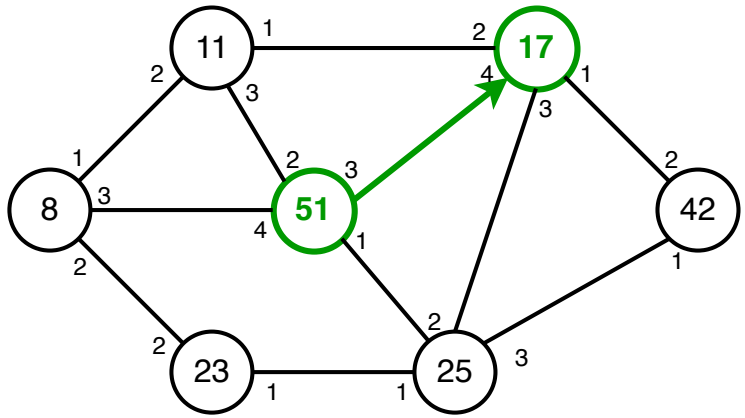
P_u is a probability distribution:

- $\forall i \in \{1, \ldots, \delta_u\}$, $P_u(i)$ gives the probability of picking i
- $P_u : \{1, \ldots, \delta_u\} \to [0, 1]$ such that $\sum_{i \in \{1, \ldots, \delta_u\}} P_u(i) = 1$
- For example, P_u may be a uniform distribution:

$$\forall i \in \{1, \ldots, \delta_u\}, P_u(i) = \frac{1}{\delta_u}$$

E.g., $P_8(1) = P_8(2) = P_8(3) = \frac{1}{3}$

Routing from 51 to 42
(*e.g.*, using uniform distribution)



Routing path: 51,17
Pick 2

Random Local Algorithm (Las Vegas Algorithm)

Given a packet p with destination label d at node u.

if $d = u$ **then**
  `deliver` p
else
  pick $i \in \{1, \ldots, \delta_u\}$ according to P_u
  send p via port number i
end if

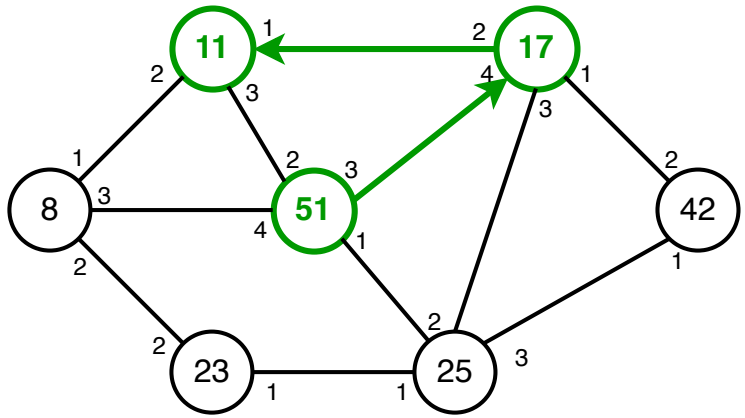
P_u is a probability distribution:

- $\forall i \in \{1, \ldots, \delta_u\}$, $P_u(i)$ gives the probability of picking i
- $P_u : \{1, \ldots, \delta_u\} \to [0, 1]$ such that $\sum_{i \in \{1, \ldots, \delta_u\}} P_u(i) = 1$
- For example, P_u may be a uniform distribution:

$$\forall i \in \{1, \ldots, \delta_u\}, P_u(i) = \frac{1}{\delta_u}$$

E.g., $P_8(1) = P_8(2) = P_8(3) = \frac{1}{3}$

Routing from 51 to 42
(*e.g.*, using uniform distribution)



Routing path: 51,17,11

Random Local Algorithm (Las Vegas Algorithm)

Given a packet p with destination label d at node u.

if $d = u$ **then**
  `deliver` p
else
  pick $i \in \{1, \ldots, \delta_u\}$ according to P_u
  send p via port number i
end if

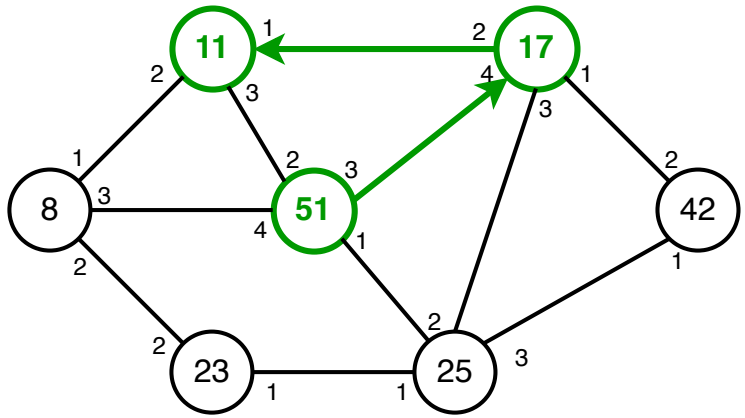
P_u is a probability distribution:

- $\forall i \in \{1, \ldots, \delta_u\}$, $P_u(i)$ gives the probability of picking i
- $P_u : \{1, \ldots, \delta_u\} \to [0, 1]$ such that $\sum_{i \in \{1, \ldots, \delta_u\}} P_u(i) = 1$
- For example, P_u may be a uniform distribution:

$$\forall i \in \{1, \ldots, \delta_u\}, P_u(i) = \frac{1}{\delta_u}$$

E.g., $P_8(1) = P_8(2) = P_8(3) = \frac{1}{3}$

Routing from 51 to 42
(*e.g.*, using uniform distribution)



Routing path: 51,17,11
Pick 3

Random Local Algorithm (Las Vegas Algorithm)

Given a packet p with destination label d at node u.

if $d = u$ **then**
  `deliver` p
else
  pick $i \in \{1, \ldots, \delta_u\}$ according to P_u
  send p via port number i
end if

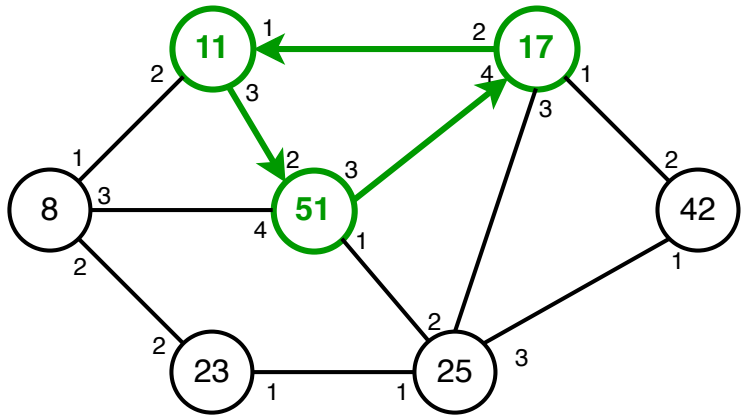
P_u is a probability distribution:

- $\forall i \in \{1, \ldots, \delta_u\}$, $P_u(i)$ gives the probability of picking i
- $P_u : \{1, \ldots, \delta_u\} \to [0, 1]$ such that $\sum_{i \in \{1, \ldots, \delta_u\}} P_u(i) = 1$
- For example, P_u may be a uniform distribution:

$$\forall i \in \{1, \ldots, \delta_u\}, P_u(i) = \frac{1}{\delta_u}$$

E.g., $P_8(1) = P_8(2) = P_8(3) = \frac{1}{3}$

Routing from 51 to 42
(*e.g.*, using uniform distribution)



Routing path: **51,17,11,51**

Random Local Algorithm (Las Vegas Algorithm)

Given a packet p with destination label d at node u.

if $d = u$ **then**
  `deliver` p
else
  pick $i \in \{1, \ldots, \delta_u\}$ according to P_u
  send p via port number i
end if

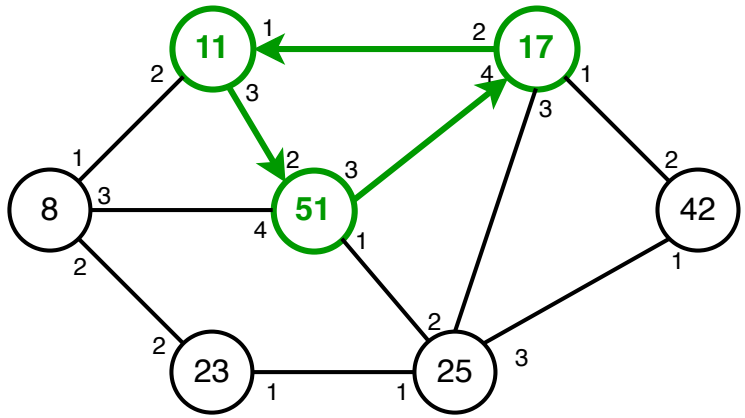
P_u is a probability distribution:

- $\forall i \in \{1, \ldots, \delta_u\}$, $P_u(i)$ gives the probability of picking i
- $P_u : \{1, \ldots, \delta_u\} \to [0, 1]$ such that $\sum_{i \in \{1, \ldots, \delta_u\}} P_u(i) = 1$
- For example, P_u may be a uniform distribution:

$$\forall i \in \{1, \ldots, \delta_u\}, P_u(i) = \frac{1}{\delta_u}$$

E.g., $P_8(1) = P_8(2) = P_8(3) = \frac{1}{3}$

Routing from 51 to 42
(*e.g.*, using uniform distribution)



Routing path: 51,17,11,51
Pick 1

Random Local Algorithm (Las Vegas Algorithm)

Given a packet p with destination label d at node u.

if $d = u$ **then**
  `deliver` p
else
  pick $i \in \{1, \ldots, \delta_u\}$ according to P_u
  send p via port number i
end if

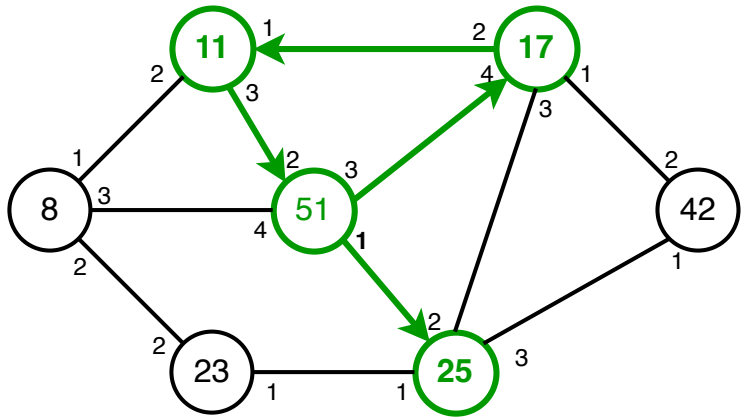
P_u is a probability distribution:

- $\forall i \in \{1, \ldots, \delta_u\}$, $P_u(i)$ gives the probability of picking i
- $P_u : \{1, \ldots, \delta_u\} \to [0, 1]$ such that $\sum_{i \in \{1, \ldots, \delta_u\}} P_u(i) = 1$
- For example, P_u may be a uniform distribution:

$$\forall i \in \{1, \ldots, \delta_u\}, P_u(i) = \frac{1}{\delta_u}$$

E.g., $P_8(1) = P_8(2) = P_8(3) = \frac{1}{3}$

Routing from 51 to 42
(*e.g.*, using uniform distribution)



Routing path: **51,17,11,51,25**

Random Local Algorithm (Las Vegas Algorithm)

Given a packet p with destination label d at node u.

if $d = u$ **then**
  `deliver` p
else
  pick $i \in \{1, \ldots, \delta_u\}$ according to P_u
  send p via port number i
end if

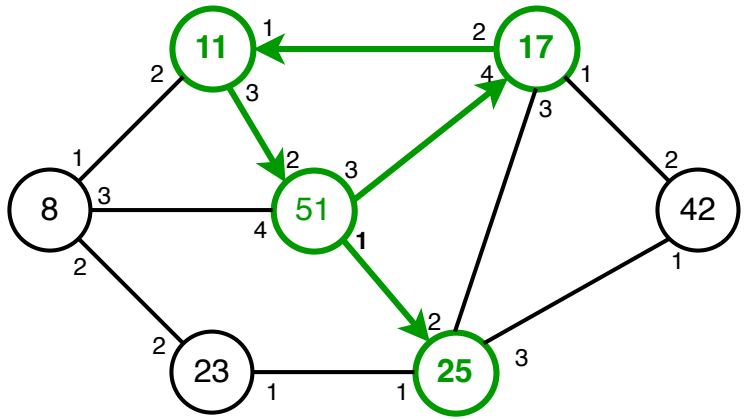
P_u is a probability distribution:

- $\forall i \in \{1, \ldots, \delta_u\}$, $P_u(i)$ gives the probability of picking i
- $P_u : \{1, \ldots, \delta_u\} \to [0, 1]$ such that $\sum_{i \in \{1, \ldots, \delta_u\}} P_u(i) = 1$
- For example, P_u may be a uniform distribution:

$$\forall i \in \{1, \ldots, \delta_u\}, P_u(i) = \frac{1}{\delta_u}$$

E.g., $P_8(1) = P_8(2) = P_8(3) = \frac{1}{3}$

Routing from 51 to 42
(*e.g.*, using uniform distribution)



Routing path: 51,17,11,51,25
Pick 3

Random Local Algorithm (Las Vegas Algorithm)

Given a packet p with destination label d at node u.

if $d = u$ **then**
  `deliver` p
else
  pick $i \in \{1, \ldots, \delta_u\}$ according to P_u
  send p via port number i
end if

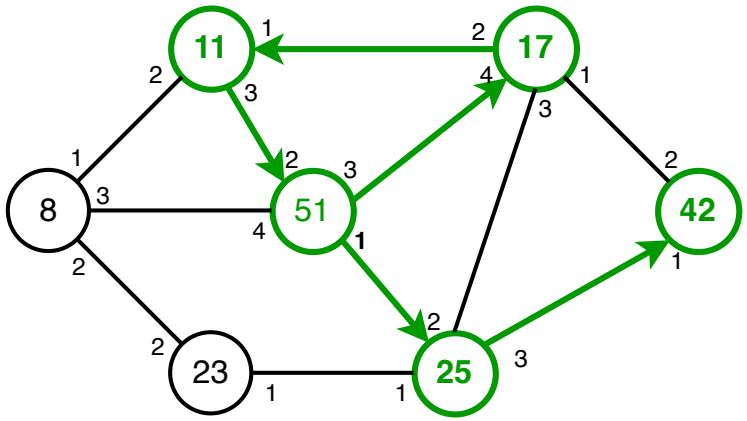
P_u is a probability distribution:

- $\forall i \in \{1, \ldots, \delta_u\}$, $P_u(i)$ gives the probability of picking i
- $P_u : \{1, \ldots, \delta_u\} \to [0, 1]$ such that $\sum_{i \in \{1, \ldots, \delta_u\}} P_u(i) = 1$
- For example, P_u may be a uniform distribution:

$$\forall i \in \{1, \ldots, \delta_u\}, P_u(i) = \frac{1}{\delta_u}$$

E.g., $P_8(1) = P_8(2) = P_8(3) = \frac{1}{3}$

Routing from 51 to 42
(*e.g.*, using uniform distribution)



Routing path: **51,17,11,51,25,42**

Random Local Algorithm (Las Vegas Algorithm)

Given a packet p with destination label d at node u.

if $d = u$ **then**
  `deliver` p
else
  pick $i \in \{1, \ldots, \delta_u\}$ according to P_u
  send p via port number i
end if

P_u is a probability distribution:

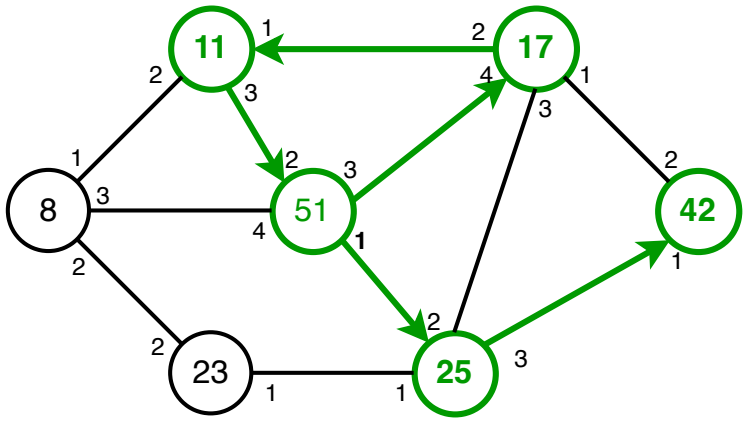
- $\forall i \in \{1, \ldots, \delta_u\}$, $P_u(i)$ gives the probability of picking i
- $P_u : \{1, \ldots, \delta_u\} \to [0, 1]$ such that $\sum_{i \in \{1, \ldots, \delta_u\}} P_u(i) = 1$
- For example, P_u may be a uniform distribution:

$$\forall i \in \{1, \ldots, \delta_u\}, P_u(i) = \frac{1}{\delta_u}$$

E.g., $P_8(1) = P_8(2) = P_8(3) = \frac{1}{3}$

- Formally, 51,17,11,51,25,42 = prefix of a **(standard) random walk**

Routing from 51 to 42
(*e.g.*, using uniform distribution)



Routing path: **51,17,11,51,25,42**

Routing using Local Information: Random Walks (also called *Drunkard's Walks*)

Réseaux & Communication

Alain Cournier Stéphane Devismes

Université de Picardie Jules Verne

October 26, 2026



Roadmap

1. Introduction
2. Correctness
3. Complexity of the Standard Random Walk
   - Relevant Quantities
   - Tool: Markov Chains
   - Hitting Time of the Standard Random Walk
   - Cover Time of the Standard Random Walk
4. Optimal (Pure) Random Walk
5. Conclusion
6. References

Roadmap

1. Introduction
2. Correctness
3. Complexity of the Standard Random Walk
   - Relevant Quantities
   - Tool: Markov Chains
   - Hitting Time of the Standard Random Walk
   - Cover Time of the Standard Random Walk
4. Optimal (Pure) Random Walk
5. Conclusion
6. References

Context

Let $G = (V, E)$ be a finite, simple, and connected graph with order $n = |V| \geq 2$ and size $m = |E|$

$\forall u \in V$, let $N(u) = \{v \mid \{u, v\} \in E\}$ be the neighborhood of u.

$N[u] = N(u) \cup \{u\}$ is the closed neighborhood of u and $\delta_u = |N(u)|$ is the degree of u

Transition Probability Matrix

We will consider **pure random walks** where the probability distribution at each node is constant[1]

The probability distributions are stored in a **transition probability matrix** P **for** G:

$$P = (p(u,v))_{u,v \in V} \in [0,1]^{V \times V}$$

- $p(u,v)$ is the probability of moving from u to v

[1] In case the probability distributions evolve along the time, a random walk is biased, *e.g.*, the simulated annealing is a biased random walk in a state space

[2] $u \in N[u]$: to be more general, we allow a walk to stay for sometime at some nodes.

Transition Probability Matrix

We will consider **pure random walks** where the probability distribution at each node is constant[1]

The probability distributions are stored in a **transition probability matrix** P **for** G:

$$P = (p(u,v))_{u,v \in V} \in [0,1]^{V \times V}$$

- $p(u,v)$ is the probability of moving from u to v
- $\forall u \in V, \sum_{v \in N[u]} p(u,v) = 1$ and $v \notin N[u] \Rightarrow p(u,v) = 0$, indeed a walk is a graph traversal[2]

[1] In case the probability distributions evolve along the time, a random walk is biased, *e.g.*, the simulated annealing is a biased random walk in a state space

[2] $u \in N[u]$: to be more general, we allow a walk to stay for sometime at some nodes.

Transition Probability Matrix

We will consider **pure random walks** where the probability distribution at each node is constant[1]

The probability distributions are stored in a **transition probability matrix P for G**:

$$P = (p(u,v))_{u,v \in V} \in [0,1]^{V \times V}$$

- $p(u,v)$ is the probability of moving from u to v
- $\forall u \in V, \sum_{v \in N[u]} p(u,v) = 1$ and $v \notin N[u] \Rightarrow p(u,v) = 0$, indeed a walk is a graph traversal[2]

Let $\mathcal{P}(G)$ be the set of all transition probability matrix for G

[1] In case the probability distributions evolve along the time, a random walk is biased, *e.g.*, the simulated annealing is a biased random walk in a state space

[2] $u \in N[u]$: to be more general, we allow a walk to stay for sometime at some nodes.

Example: Uniform Transition Probability Matrix

$\forall u, v \in V$:

- $v \notin N(u) \Rightarrow p(u, v) = 0$
- $v \in N(u) \Rightarrow p(u, v) = \frac{1}{\delta_u}$

Remark: $\forall u, p(u, u) = 0$, so no wait!

Random Walk

Definition

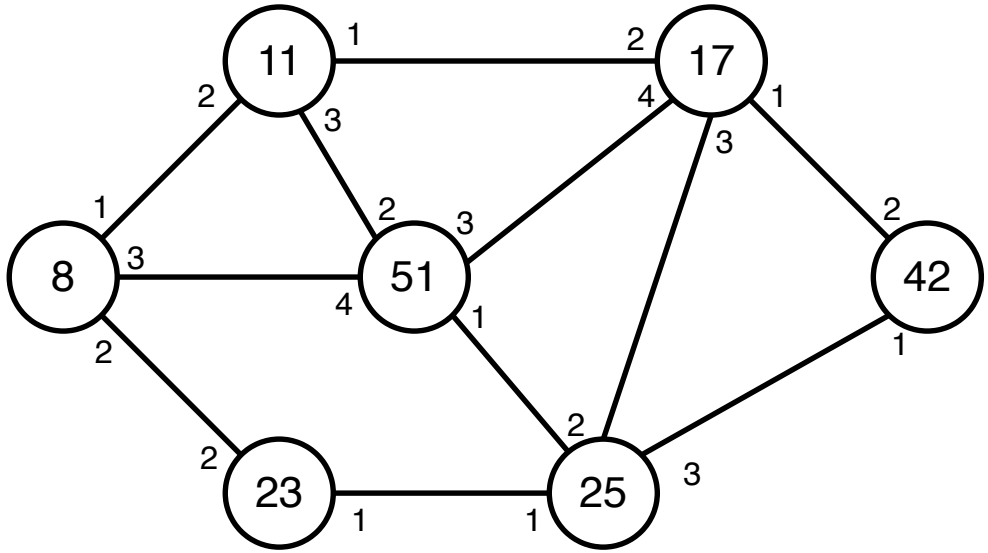
A **random walk** $\omega = (\omega_0, \omega_1, \ldots)$ on G starting at vertex u under $P \in \mathcal{P}(G)$ is an infinite sequence of random variables ω_i whose domain is V such that

- $\omega_0 = u$ with probability 1, and
- $\forall i \in \mathbb{N}$, the (conditional) probability that $\omega_{i+1} = w$, provided that $\omega_i = v$, is $p(v, w)$

Random Walk

Example

The infinite random sequence $(51, 17, 11)^\omega$ is a random walk on the graph given below under a uniform transition probability matrix.



Remark: a random walk on a graph under a uniform transition probability matrix is called a **standard random walk**

Roadmap

1. Introduction
2. Correctness
3. Complexity of the Standard Random Walk
   - Relevant Quantities
   - Tool: Markov Chains
   - Hitting Time of the Standard Random Walk
   - Cover Time of the Standard Random Walk
4. Optimal (Pure) Random Walk
5. Conclusion
6. References

Correctness of the Random-walk-based Algorithm

Correctness = Partial Correctness + Termination

Partial Correctness: Trivial! The algorithm stops only if the packet has reached its destination.

Termination *almost sure*: Termination with probability one (Las Vegas Algorithm).

I.e., there are infinite executions where the destination is never reached (*e.g.*, $(51, 17, 11)^\omega$), yet the overall probability of the occurrence of such executions is 0.

The almost sure termination is due to the fact that any vertex has probability 1 of occurring in any standard random walk on G.

Characterization

Let $S = (V_S, E_S)$ be the digraph such that

- $V_S = V$ and
- $E_S = \{(u, v) \in V^2 \mid p(u, v) > 0\}$

Theorem 1

For every $u, v \in V$, v has probability 1 of occurring in any random walk on G starting at vertex u under $P \in \mathcal{P}(G)$

if and only if

S is strongly connected.

Corollary 2

v has probability 1 of occurring in any standard random walk on G.

Proof of Theorem 1

Necessary Condition

Assume S is not strongly connected and let u, v be two nodes of S such that v is not reachable from u. $(u \neq v)$

Proof of Theorem 1

Necessary Condition

Assume S is not strongly connected and let u, v be two nodes of S such that v is not reachable from u. $(u \neq v)$

Assume, by the contradiction, that v occurs in a random walk on G starting at vertex u under P.

Let SP by the smallest prefix of the walk starting from u and ending with v.

Proof of Theorem 1

Necessary Condition

Assume S is not strongly connected and let u, v be two nodes of S such that v is not reachable from u. $(u \neq v)$

Assume, by the contradiction, that v occurs in a random walk on G starting at vertex u under P.

Let SP by the smallest prefix of the walk starting from u and ending with v.

Every two consecutive nodes w and w' in SP satisfies $p(w, w') > 0$, which in turn implies $(w, w') \in E_S$.

SP is a (directed) path from u to v in S: v is reachable from u in S, a contradiction. □

Proof of Theorem 1

Sufficient Condition

Let ω be any random walk on G starting at vertex u under P. Let $p_{\min} = \min\{p(w, w') \mid (w, w') \in E_S\}$.

Proof of Theorem 1

Sufficient Condition

Let ω be any random walk on G starting at vertex u under P. Let $p_{\min} = \min\{p(w, w') \mid (w, w') \in E_S\}$.

$p_{\min} > 0$, by definition of S.

Proof of Theorem 1

Sufficient Condition

Let ω be any random walk on G starting at vertex u under P. Let $p_{\min} = \min\{p(w, w') \mid (w, w') \in E_S\}$.

$p_{\min} > 0$, by definition of S.

Since S is strongly connected and $n > 1$, its diameter $\mathcal{D}$ satisfies $1 \leq \mathcal{D} < n$.

Proof of Theorem 1

Sufficient Condition

Let ω be any random walk on G starting at vertex u under P. Let $p_{\min} = \min\{p(w, w') \mid (w, w') \in E_S\}$.

$p_{\min} > 0$, by definition of S.

Since S is strongly connected and $n > 1$, its diameter $\mathcal{D}$ satisfies $1 \leq \mathcal{D} < n$.

In every suffix s of ω, the probability that v occurs among the first $\mathcal{D}$ values of s is at least $0 < (p_{\min})^{\mathcal{D}} \leq 1$. Indeed, there is an elementary path of length at most $\mathcal{D}$ from any vertex to v in S.

Proof of Theorem 1

Sufficient Condition

Let ω be any random walk on G starting at vertex u under P. Let $p_{\min} = \min\{p(w, w') \mid (w, w') \in E_S\}$.

$p_{\min} > 0$, by definition of S.

Since S is strongly connected and $n > 1$, its diameter $\mathcal{D}$ satisfies $1 \leq \mathcal{D} < n$.

In every suffix s of ω, the probability that v occurs among the first $\mathcal{D}$ values of s is at least $0 < (p_{\min})^{\mathcal{D}} \leq 1$. Indeed, there is an elementary path of length at most $\mathcal{D}$ from any vertex to v in S.

So, the probability that v does not occur among the first $k \times \mathcal{D}$ values of ω is at most $(1 - (p_{\min})^{\mathcal{D}})^k$.

Proof of Theorem 1

Sufficient Condition

Let ω be any random walk on G starting at vertex u under P. Let $p_{\min} = \min\{p(w, w') \mid (w, w') \in E_S\}$.

$p_{\min} > 0$, by definition of S.

Since S is strongly connected and $n > 1$, its diameter $\mathcal{D}$ satisfies $1 \leq \mathcal{D} < n$.

In every suffix s of ω, the probability that v occurs among the first $\mathcal{D}$ values of s is at least $0 < (p_{\min})^{\mathcal{D}} \leq 1$. Indeed, there is an elementary path of length at most $\mathcal{D}$ from any vertex to v in S.

So, the probability that v does not occur among the first $k \times \mathcal{D}$ values of ω is at most $(1 - (p_{\min})^{\mathcal{D}})^k$.

Now, $\lim_{k\to\infty}(1 - (p_{\min})^{\mathcal{D}})^k = 0$ since $0 \leq 1 - (p_{\min})^{\mathcal{D}} < 1$.

Hence, v has probability 1 of occurring in ω. □

Roadmap

1. Introduction
2. Correctness
3. Complexity of the Standard Random Walk
   - Relevant Quantities
   - Tool: Markov Chains
   - Hitting Time of the Standard Random Walk
   - Cover Time of the Standard Random Walk
4. Optimal (Pure) Random Walk
5. Conclusion
6. References

Roadmap

1. Introduction
2. Correctness
3. Complexity of the Standard Random Walk
   - Relevant Quantities
   - Tool: Markov Chains
   - Hitting Time of the Standard Random Walk
   - Cover Time of the Standard Random Walk
4. Optimal (Pure) Random Walk
5. Conclusion
6. References

Relevant Quantities

Hitting Time: informally, the hitting time is the expected time to move to a node v in a random walk

(from a routing point of view, it is the expected length of the routing path)

Cover Time: informally, the cover time is the expected time to visit all nodes in a random walk

Hitting Time

Given a random walk $\omega = (\omega_0, \omega_1, \ldots)$ starting at vertex $u \in V$, the hitting time $H_G(P; u, v)$ from u to v under P is:

$$H_G(P; u, v) = E_P[\inf\{i \geq 1 \mid \omega_i = v\}]$$

i.e., the expectation of the smallest time where ω reaches v after leaving u.

Hitting Time

Given a random walk $\omega = (\omega_0, \omega_1, \ldots)$ starting at vertex $u \in V$, the hitting time $H_G(P; u, v)$ from u to v under P is:

$$H_G(P; u, v) = E_P[\inf\{i \geq 1 \mid \omega_i = v\}]$$

i.e., the expectation of the smallest time where ω reaches v after leaving u.

Remark: $H_G(P; u, u)$ is the expectation of the smallest time for ω to leave and then return to u!

The hitting time $H_G(P)$ of G under P is:

$$H_G(P) = \max_{u,v \in V} H_G(P; u, v)$$

Hitting Time

Given a random walk $\omega = (\omega_0, \omega_1, \ldots)$ starting at vertex $u \in V$, the hitting time $H_G(P; u, v)$ from u to v under P is:

$$H_G(P; u, v) = E_P[\inf\{i \geq 1 \mid \omega_i = v\}]$$

i.e., the expectation of the smallest time where ω reaches v after leaving u.

Remark: $H_G(P; u, u)$ is the expectation of the smallest time for ω to leave and then return to u!

The hitting time $H_G(P)$ of G under P is:

$$H_G(P) = \max_{u,v \in V} H_G(P; u, v)$$

In the following, we will denote by $\mathcal{H}_G(u, v)$ the hitting time from u to v in a standard random walk on G.

Cover Time

Given a random walk $\omega = (\omega_0, \omega_1, \ldots)$ starting at vertex $u \in V$, the cover time $C_G(P; u)$ from u under P is:

$$C_G(P; u) = E_P[\inf\{i \geq 1 \mid \{\omega_0, \ldots, \omega_i\} = V\}]$$

i.e., the expectation of the smallest time for ω to visit all vertices starting from u.

The cover time $C_G(P)$ of G under P is:

$$C_G(P) = \max_{u \in V} C_G(P; u)$$

Roadmap

1. Introduction
2. Correctness
3. Complexity of the Standard Random Walk
   - Relevant Quantities
   - Tool: Markov Chains
   - Hitting Time of the Standard Random Walk
   - Cover Time of the Standard Random Walk
4. Optimal (Pure) Random Walk
5. Conclusion
6. References

Definition

A **Markov chain** or Markov process is a stochastic model where the probability of future (next) state only depends on the most recent (current) state.

This memoryless property of a stochastic process is called Markov property.

From a probability perspective, the Markov property implies that the conditional probability distribution of the future state (conditioned on both past and current states) only depends on the current state.

Definition

A **Markov chain** or Markov process is a stochastic model where the probability of future (next) state only depends on the most recent (current) state.

This memoryless property of a stochastic process is called Markov property.

From a probability perspective, the Markov property implies that the conditional probability distribution of the future state (conditioned on both past and current states) only depends on the current state.

A Markov chain is usually represented as a weighted digraph where nodes are states and arcs are possible transitions weighted with their positive probability of occurrence.

Definition

A **Markov chain** or Markov process is a stochastic model where the probability of future (next) state only depends on the most recent (current) state.

This memoryless property of a stochastic process is called Markov property.

From a probability perspective, the Markov property implies that the conditional probability distribution of the future state (conditioned on both past and current states) only depends on the current state.

A Markov chain is usually represented as a weighted digraph where nodes are states and arcs are possible transitions weighted with their positive probability of occurrence.

A Markov chain in which every state can be reached from every other state is called an **irreducible Markov chain**.

Example

A random walk on a graph as the Markov property: it can be modeled by a finite Markov chain.

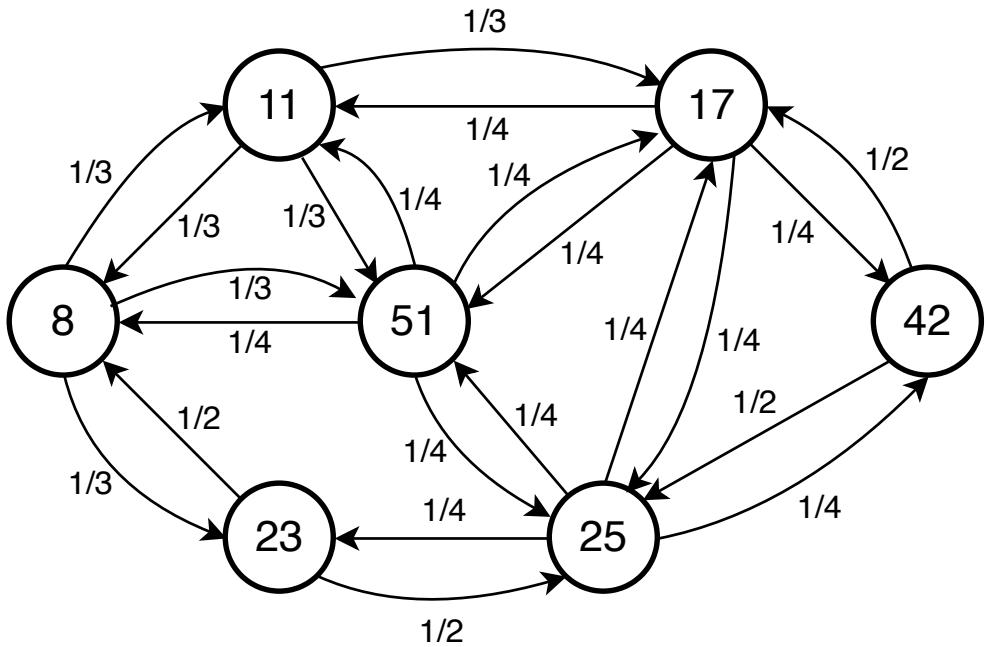
For example, the weighted digraph S in Theorem 1 is a Markov Chain.

Example

A random walk on a graph as the Markov property: it can be modeled by a finite Markov chain.

For example, the weighted digraph S in Theorem 1 is a Markov Chain.

Below, we give the Markov chain corresponding to the standard random walk on our sample graph.



Stationary Distribution of a Markov Chain

The **stationary distribution** $\pi = (\pi_i)_{i \in E}$ of a Markov chain gives the fraction of the time spent in each state i of the state space E of this Markov chain, asymptotically.

Let $S_n(i)$ the time spent in state i after the first n steps.

$$\pi_i = \lim_{n \to \infty} \frac{S_n(i)}{n}$$

Application of the theorem of Perron-Frobenius [5, 8]

Corollary 3

Any ***finite irreducible Markov chain*** *has a stationary distribution* $\pi = (\pi_i)_{i \in E}$ *that is the* ***unique solution*** *of:*

1. $\sum_{i \in E} \pi_i = 1$, *and*
2. $\forall j \in E, \sum_{i \in E} \pi_i p(i,j) = \pi_j$

where $p(i,j)$ *are the transition probabilities of the Markov chain.*

Intuition:

Application of the theorem of Perron-Frobenius [5, 8]

Corollary 3

*Any **finite irreducible Markov chain** has a stationary distribution $\pi = (\pi_i)_{i \in E}$ that is the **unique solution** of:*

1. $\sum_{i \in E} \pi_i = 1$, *and*
2. $\forall j \in E, \sum_{i \in E} \pi_i p(i,j) = \pi_j$

where $p(i,j)$ are the transition probabilities of the Markov chain.

Intuition:

1. In a distribution, the sum of probabilities is equal to 1

Application of the theorem of Perron-Frobenius [5, 8]

Corollary 3

*Any **finite irreducible Markov chain** has a stationary distribution $\pi = (\pi_i)_{i \in E}$ that is the **unique solution** of:*

1. $\sum_{i \in E} \pi_i = 1$, *and*
2. $\forall j \in E, \sum_{i \in E} \pi_i p(i,j) = \pi_j$

where $p(i,j)$ are the transition probabilities of the Markov chain.

Intuition:

1. In a distribution, the sum of probabilities is equal to 1
2. From i, j is reached in one step with probability $p(i,j)$: it is the fraction of time j is reached from i provided that the walk is in i

Application of the theorem of Perron-Frobenius [5, 8]

Corollary 3

Any ***finite irreducible Markov chain*** *has a stationary distribution* $\pi = (\pi_i)_{i \in E}$ *that is the* ***unique solution*** *of:*

1. $\sum_{i \in E} \pi_i = 1$, *and*
2. $\forall j \in E, \sum_{i \in E} \pi_i p(i,j) = \pi_j$

where $p(i,j)$ *are the transition probabilities of the Markov chain.*

Intuition:

1. In a distribution, the sum of probabilities is equal to 1
2. From i, j is reached in one step with probability $p(i,j)$: it is the fraction of time j is reached from i provided that the walk is in i

   π_i is the fraction of the time spent in i

   So, $\pi_i p(i,j)$ gives the fraction of time j is reached from i

Application of the theorem of Perron-Frobenius [5, 8]

Corollary 3

Any ***finite irreducible Markov chain*** *has a stationary distribution* $\pi = (\pi_i)_{i \in E}$ *that is the* ***unique solution*** *of:*

1. $\sum_{i \in E} \pi_i = 1$, *and*
2. $\forall j \in E, \sum_{i \in E} \pi_i p(i,j) = \pi_j$

where $p(i,j)$ *are the transition probabilities of the Markov chain.*

Intuition:

1. In a distribution, the sum of probabilities is equal to 1
2. From i, j is reached in one step with probability $p(i,j)$: it is the fraction of time j is reached from i provided that the walk is in i

   π_i is the fraction of the time spent in i

   So, $\pi_i p(i,j)$ gives the fraction of time j is reached from i

   Now, the fraction of the time spent in each state j, π_j, is the fraction of time j is reached from all states of E

Fundamental Result

from [6]

Lemma 4

$\forall u \in V, \mathcal{H}_G(u,u) = \frac{2m}{\delta_u}$

Intuition:

Fundamental Result

from [6]

Lemma 4

$\forall u \in V, \mathcal{H}_G(u,u) = \frac{2m}{\delta_u}$

Intuition:

- Since G has m edges, the Markov chain associated to the standard random walk on G has $2m$ arcs

Fundamental Result

from [6]

Lemma 4

$\forall u \in V, \mathcal{H}_G(u,u) = \frac{2m}{\delta_u}$

Intuition:

- Since G has m edges, the Markov chain associated to the standard random walk on G has $2m$ arcs
- Since the random walk is standard, the traversing of any arc is asymptotically equiprobable, *i.e.*, the stationary probability of any arc is $\frac{1}{2m}$

Fundamental Result

from [6]

Lemma 4

$\forall u \in V, \mathcal{H}_G(u,u) = \frac{2m}{\delta_u}$

Intuition:

- Since G has m edges, the Markov chain associated to the standard random walk on G has $2m$ arcs
- Since the random walk is standard, the traversing of any arc is asymptotically equiprobable, *i.e.*, the stationary probability of any arc is $\frac{1}{2m}$
- The stationary probability of a node u, π_u, is the sum of the stationary probability of its incoming arcs

Fundamental Result

from [6]

Lemma 4

$\forall u \in V, \mathcal{H}_G(u,u) = \frac{2m}{\delta_u}$

Intuition:

- Since G has m edges, the Markov chain associated to the standard random walk on G has $2m$ arcs
- Since the random walk is standard, the traversing of any arc is asymptotically equiprobable, *i.e.*, the stationary probability of any arc is $\frac{1}{2m}$
- The stationary probability of a node u, π_u, is the sum of the stationary probability of its incoming arcs
- Since a node u has δ_u incoming arcs in the Markov chain, we have $\pi_u = \frac{\delta_u}{2m}$

Fundamental Result

from [6]

Lemma 4

$\forall u \in V, \mathcal{H}_G(u,u) = \frac{2m}{\delta_u}$

Intuition:

- Since G has m edges, the Markov chain associated to the standard random walk on G has $2m$ arcs
- Since the random walk is standard, the traversing of any arc is asymptotically equiprobable, *i.e.*, the stationary probability of any arc is $\frac{1}{2m}$
- The stationary probability of a node u, π_u, is the sum of the stationary probability of its incoming arcs
- Since a node u has δ_u incoming arcs in the Markov chain, we have $\pi_u = \frac{\delta_u}{2m}$
- Since π_u is the fraction of the time spent in vertex u during the walk, we have $\mathcal{H}_G(u,u) = \frac{1}{\pi_u}$, *i.e.*, $\mathcal{H}_G(u,u) = \frac{2m}{\delta_u}$

Proof of Lemma 4 (1/2)

Consider an arbitrary standard random walk ω on G. Let $\pi = (\pi_v)_{v \in V}$ be the stationary distribution of Markov chain that models ω.

Proof of Lemma 4 (1/2)

Consider an arbitrary standard random walk ω on G. Let $\pi = (\pi_v)_{v \in V}$ be the stationary distribution of Markov chain that models ω.

$\mathcal{H}_G(u, u) = \frac{1}{\pi_u}$.

Proof of Lemma 4 (1/2)

Consider an arbitrary standard random walk ω on G. Let $\pi = (\pi_v)_{v \in V}$ be the stationary distribution of Markov chain that models ω.

$\mathcal{H}_G(u, u) = \frac{1}{\pi_u}$.

Thus, the lemma holds if $\pi_u = \frac{\delta_u}{2m}$.

Proof of Lemma 4 (1/2)

Consider an arbitrary standard random walk ω on G. Let $\pi = (\pi_v)_{v \in V}$ be the stationary distribution of Markov chain that models ω.

$\mathcal{H}_G(u, u) = \frac{1}{\pi_u}$.

Thus, the lemma holds if $\pi_u = \frac{\delta_u}{2m}$.

V is a finite set. G is connected and at each node, the probability of traversing each incident edge is strictly positive. So, the Markov chain modeling ω is finite and irreducible. Hence, Corollary 3 applies: $\forall u \in V$, $\pi_u := \frac{\delta_u}{2m}$ should be the solution of

- $\sum_{u \in V} \pi_u = 1$, and
- $\forall v \in V$, $\sum_{u \in V} \pi_u p(u, v) = \pi_v$

where $p(u, v) = \frac{1}{\delta_u}$ if u and v are neighbors, 0 otherwise.

Proof of Lemma 4 (1/2)

Consider an arbitrary standard random walk ω on G. Let $\pi = (\pi_v)_{v \in V}$ be the stationary distribution of Markov chain that models ω.

$\mathcal{H}_G(u, u) = \frac{1}{\pi_u}$.

Thus, the lemma holds if $\pi_u = \frac{\delta_u}{2m}$.

V is a finite set. G is connected and at each node, the probability of traversing each incident edge is strictly positive. So, the Markov chain modeling ω is finite and irreducible. Hence, Corollary 3 applies: $\forall u \in V$, $\pi_u := \frac{\delta_u}{2m}$ should be the solution of

- $\sum_{u \in V} \pi_u = 1$, and
- $\forall v \in V$, $\sum_{u \in V} \pi_u p(u, v) = \pi_v$

where $p(u, v) = \frac{1}{\delta_u}$ if u and v are neighbors, 0 otherwise.

$$\sum_{u \in V} \pi_u = \sum_{u \in V} \frac{\delta_u}{2m}$$

Proof of Lemma 4 (1/2)

Consider an arbitrary standard random walk ω on G. Let $\pi = (\pi_v)_{v \in V}$ be the stationary distribution of Markov chain that models ω.

$\mathcal{H}_G(u, u) = \frac{1}{\pi_u}$.

Thus, the lemma holds if $\pi_u = \frac{\delta_u}{2m}$.

V is a finite set. G is connected and at each node, the probability of traversing each incident edge is strictly positive. So, the Markov chain modeling ω is finite and irreducible. Hence, Corollary 3 applies: $\forall u \in V$, $\pi_u := \frac{\delta_u}{2m}$ should be the solution of

- $\sum_{u \in V} \pi_u = 1$, and
- $\forall v \in V$, $\sum_{u \in V} \pi_u p(u, v) = \pi_v$

where $p(u, v) = \frac{1}{\delta_u}$ if u and v are neighbors, 0 otherwise.

$$\sum_{u \in V} \pi_u = \sum_{u \in V} \frac{\delta_u}{2m}$$
$$= \frac{\sum_{u \in V} \delta_u}{2m}$$

Proof of Lemma 4 (1/2)

Consider an arbitrary standard random walk ω on G. Let $\pi = (\pi_v)_{v \in V}$ be the stationary distribution of Markov chain that models ω.

$\mathcal{H}_G(u, u) = \frac{1}{\pi_u}$.

Thus, the lemma holds if $\pi_u = \frac{\delta_u}{2m}$.

V is a finite set. G is connected and at each node, the probability of traversing each incident edge is strictly positive. So, the Markov chain modeling ω is finite and irreducible. Hence, Corollary 3 applies: $\forall u \in V$, $\pi_u := \frac{\delta_u}{2m}$ should be the solution of

- $\sum_{u \in V} \pi_u = 1$, and
- $\forall v \in V$, $\sum_{u \in V} \pi_u p(u, v) = \pi_v$

where $p(u, v) = \frac{1}{\delta_u}$ if u and v are neighbors, 0 otherwise.

$$\begin{aligned} \sum_{u \in V} \pi_u &= \sum_{u \in V} \frac{\delta_u}{2m} \\ &= \frac{\sum_{u \in V} \delta_u}{2m} \\ &= \frac{2m}{2m} = 1 \end{aligned}$$

(handshaking lemma, [2])

Proof of Lemma 4 (2/2)

Let $v \in V$.

$$\sum_{u \in V} \pi_u p(u, v) = \sum_{u \in N(v)} \frac{\pi_u}{\delta_u}$$

standard random walk

Proof of Lemma 4 (2/2)

Let $v \in V$.

$$\sum_{u \in V} \pi_u p(u,v) = \sum_{u \in N(v)} \frac{\pi_u}{\delta_u}$$

$$= \sum_{u \in N(v)} \frac{\delta_u}{\delta_u 2m}$$

standard random walk

$$\pi_u := \frac{\delta_u}{2m}$$

Proof of Lemma 4 (2/2)

Let $v \in V$.

$$\sum_{u \in V} \pi_u p(u,v) = \sum_{u \in N(v)} \frac{\pi_u}{\delta_u}$$

$$= \sum_{u \in N(v)} \frac{\delta_u}{\delta_u 2m}$$

$$= \sum_{u \in N(v)} \frac{1}{2m}$$

standard random walk

$$\pi_u := \frac{\delta_u}{2m}$$

Proof of Lemma 4 (2/2)

Let $v \in V$.

$$\sum_{u \in V} \pi_u p(u,v) = \sum_{u \in N(v)} \frac{\pi_u}{\delta_u} \qquad \text{standard random walk}$$

$$= \sum_{u \in N(v)} \frac{\delta_u}{\delta_u 2m} \qquad \pi_u := \frac{\delta_u}{2m}$$

$$= \sum_{u \in N(v)} \frac{1}{2m}$$

$$= \frac{\delta_v}{2m} \qquad \delta_v = |N(v)|$$

$$= \pi_v$$

Thus, $\forall u \in V$, $\pi_u := \frac{\delta_u}{2m}$ is the solution! □

Roadmap

1. Introduction
2. Correctness
3. Complexity of the Standard Random Walk
   - Relevant Quantities
   - Tool: Markov Chains
   - Hitting Time of the Standard Random Walk
   - Cover Time of the Standard Random Walk
4. Optimal (Pure) Random Walk
5. Conclusion
6. References

Bounds

In [1], the hitting time of the standard random walk is shown to be in

$$\Theta(n^3)$$

Bounds

In [1], the hitting time of the standard random walk is shown to be in

$$\Theta(n^3)$$

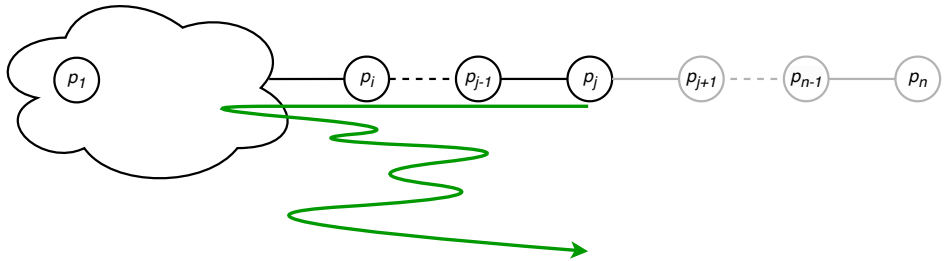
Let us now study the worst case

Basic properties

Let $p_1, \ldots, p_n$ the vertices of V.

Assume G has a pending line $L = p_i, \ldots, p_n$ with $i > 1$: $\forall j \in \{i, \ldots, n-1\}, \delta_{p_j} = 2$, $\delta_{p_n} = 1$, and the subgraph $G(L)$ induced by L is a line. Let p_{i-1} the neighbor of p_i such that $p_{i-1} \notin L$.

Assume a random walk starting from p_1

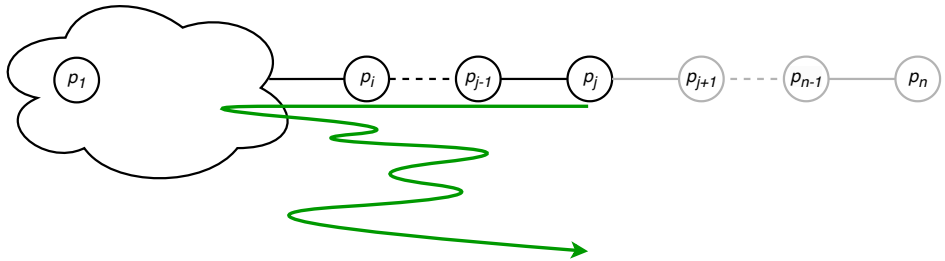


Basic properties

Let $p_1, \ldots, p_n$ the vertices of V.

Assume G has a pending line $L = p_i, \ldots, p_n$ with $i > 1$: $\forall j \in \{i, \ldots, n-1\}, \delta_{p_j} = 2$, $\delta_{p_n} = 1$, and the subgraph $G(L)$ induced by L is a line. Let p_{i-1} the neighbor of p_i such that $p_{i-1} \notin L$.

Assume a random walk starting from p_1



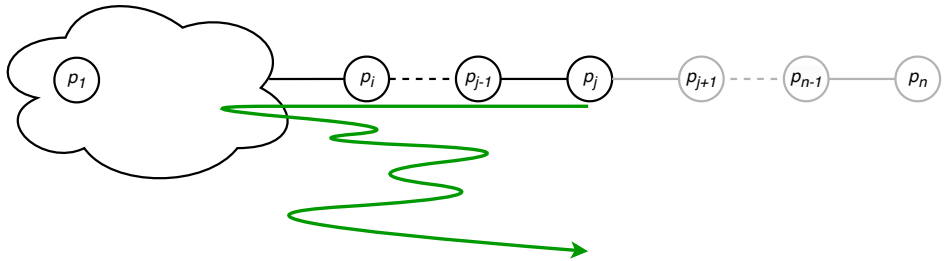
1. A walk that leaves and returns to p_n necessarily first goes to p_{n-1}, so $\mathcal{H}_G(p_n, p_n) = \mathcal{H}_G(p_n, p_{n-1}) + \mathcal{H}_G(p_{n-1}, p_n) = 1 + \mathcal{H}_G(p_{n-1}, p_n)$, so $\mathcal{H}_G(p_{n-1}, p_n) = \mathcal{H}_G(p_n, p_n) - 1 = 2m - 1$, by Lemma 4

Basic properties

Let $p_1, \ldots, p_n$ the vertices of V.

Assume G has a pending line $L = p_i, \ldots, p_n$ with $i > 1$: $\forall j \in \{i, \ldots, n-1\}, \delta_{p_j} = 2$, $\delta_{p_n} = 1$, and the subgraph $G(L)$ induced by L is a line. Let p_{i-1} the neighbor of p_i such that $p_{i-1} \notin L$.

Assume a random walk starting from p_1



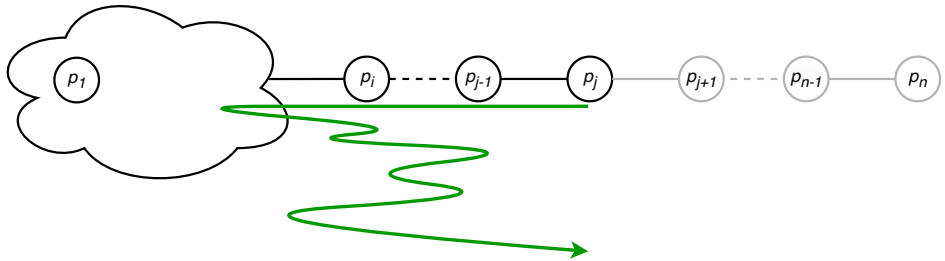
1. A walk that leaves and returns to p_n necessarily first goes to p_{n-1}, so $\mathcal{H}_G(p_n, p_n) = \mathcal{H}_G(p_n, p_{n-1}) + \mathcal{H}_G(p_{n-1}, p_n) = 1 + \mathcal{H}_G(p_{n-1}, p_n)$, so $\mathcal{H}_G(p_{n-1}, p_n) = \mathcal{H}_G(p_n, p_n) - 1 = 2m - 1$, by Lemma 4
2. $\forall j \in \{i, \ldots, n\}$, $\mathcal{H}_G(p_1, p_j) = \mathcal{H}_G(p_1, p_{j-1}) + \mathcal{H}_G(p_{j-1}, p_j)$: a walk from p_1 to p_j necessarily go via p_{j-1}

Basic properties

Let $p_1, \ldots, p_n$ the vertices of V.

Assume G has a pending line $L = p_i, \ldots, p_n$ with $i > 1$: $\forall j \in \{i, \ldots, n-1\}, \delta_{p_j} = 2$, $\delta_{p_n} = 1$, and the subgraph $G(L)$ induced by L is a line. Let p_{i-1} the neighbor of p_i such that $p_{i-1} \notin L$.

Assume a random walk starting from p_1



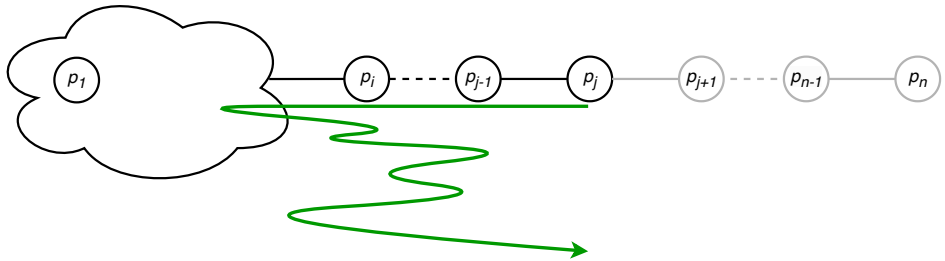
1. A walk that leaves and returns to p_n necessarily first goes to p_{n-1}, so $\mathcal{H}_G(p_n, p_n) = \mathcal{H}_G(p_n, p_{n-1}) + \mathcal{H}_G(p_{n-1}, p_n) = 1 + \mathcal{H}_G(p_{n-1}, p_n)$, so $\mathcal{H}_G(p_{n-1}, p_n) = \mathcal{H}_G(p_n, p_n) - 1 = 2m - 1$, by Lemma 4
2. $\forall j \in \{i, \ldots, n\}$, $\mathcal{H}_G(p_1, p_j) = \mathcal{H}_G(p_1, p_{j-1}) + \mathcal{H}_G(p_{j-1}, p_j)$: a walk from p_1 to p_j necessarily go via p_{j-1}
3. $\forall j \in \{i, \ldots, n\}$, $\mathcal{H}_G(p_{j-1}, p_j) = \mathcal{P}_{G(V \setminus \{p_{j+1}, \ldots, p_n\})}(p_{j-1}, p_j)$: a walk from p_{j-1} hits p_j before any vertex in $p_{j+1}, \ldots, p_n$.

Basic properties

Let $p_1, \ldots, p_n$ the vertices of V.

Assume G has a pending line $L = p_i, \ldots, p_n$ with $i > 1$: $\forall j \in \{i, \ldots, n-1\}, \delta_{p_j} = 2$, $\delta_{p_n} = 1$, and the subgraph $G(L)$ induced by L is a line. Let p_{i-1} the neighbor of p_i such that $p_{i-1} \notin L$.

Assume a random walk starting from p_1



1. A walk that leaves and returns to p_n necessarily first goes to p_{n-1}, so $\mathcal{H}_G(p_n, p_n) = \mathcal{H}_G(p_n, p_{n-1}) + \mathcal{H}_G(p_{n-1}, p_n) = 1 + \mathcal{H}_G(p_{n-1}, p_n)$, so $\mathcal{H}_G(p_{n-1}, p_n) = \mathcal{H}_G(p_n, p_n) - 1 = 2m - 1$, by Lemma 4
2. $\forall j \in \{i, \ldots, n\}$, $\mathcal{H}_G(p_1, p_j) = \mathcal{H}_G(p_1, p_{j-1}) + \mathcal{H}_G(p_{j-1}, p_j)$: a walk from p_1 to p_j necessarily go via p_{j-1}
3. $\forall j \in \{i, \ldots, n\}$, $\mathcal{H}_G(p_{j-1}, p_j) = \mathcal{P}_{G(V \setminus \{p_{j+1}, \ldots, p_n\})}(p_{j-1}, p_j)$: a walk from p_{j-1} hits p_j before any vertex in $p_{j+1}, \ldots, p_n$.
4. $\forall j \in \{i, \ldots, n\}$, $\mathcal{H}_G(p_{j-1}, p_j) = \mathcal{P}_{G(V \setminus \{p_{j+1}, \ldots, p_n\})}(p_{j-1}, p_j) = 2(m - (n - j)) - 1 = 2m - (2n - 2j + 1)$ by Property 1

First attempt: a line L

L: p_1 — p_2 — . . . — p_n with $n > 1$

L is p_1 linked to a pending line p_2, . . ., p_n so previous properties apply with $i = 2$.

$$\mathcal{H}_G(p_1, p_n) =$$

First attempt: a line L

L: p_1 — p_2 — ... — p_n with $n > 1$

L is p_1 linked to a pending line p_2, ..., p_n so previous properties apply with $i = 2$.

$$\mathcal{H}_G(p_1, p_n) = \sum_{j=2}^{n} \mathcal{H}_G(p_{j-1}, p_j) \qquad \text{by Property 2}$$

First attempt: a line L

L: p_1 — p_2 — ... — p_n with $n > 1$

L is p_1 linked to a pending line p_2, ..., p_n so previous properties apply with $i = 2$.

$$\mathcal{H}_G(p_1, p_n) = \sum_{j=2}^{n} \mathcal{H}_G(p_{j-1}, p_j) \qquad \text{by Property 2}$$

$$= \sum_{j=2}^{n} (2m - (2n - 2j + 1)) \qquad \text{by Property 4}$$

First attempt: a line L

L: p_1 — p_2 — ... — p_n with $n > 1$

L is p_1 linked to a pending line p_2, ..., p_n so previous properties apply with $i = 2$.

$$\mathcal{H}_G(p_1, p_n) = \sum_{j=2}^{n} \mathcal{H}_G(p_{j-1}, p_j) \qquad \text{by Property 2}$$

$$= \sum_{j=2}^{n} (2m - (2n - 2j + 1)) \qquad \text{by Property 4}$$

$$= \sum_{j=2}^{n} (2n - 2 - (2n - 2j + 1)) \qquad m = n - 1$$

First attempt: a line L

L: p_1 — p_2 — ... — p_n with $n > 1$

L is p_1 linked to a pending line p_2, ..., p_n so previous properties apply with $i = 2$.

$$\mathcal{H}_G(p_1, p_n) = \sum_{j=2}^{n} \mathcal{H}_G(p_{j-1}, p_j) \qquad \text{by Property 2}$$

$$= \sum_{j=2}^{n} (2m - (2n - 2j + 1)) \qquad \text{by Property 4}$$

$$= \sum_{j=2}^{n} (2n - 2 - (2n - 2j + 1)) \qquad m = n - 1$$

$$= \sum_{j=2}^{n} (2j - 3)$$

First attempt: a line L

L: p_1 — p_2 — ... — p_n with $n > 1$

L is p_1 linked to a pending line p_2, ..., p_n so previous properties apply with $i = 2$.

$$\mathcal{H}_G(p_1, p_n) = \sum_{j=2}^{n} \mathcal{H}_G(p_{j-1}, p_j) \qquad \text{by Property 2}$$

$$= \sum_{j=2}^{n} (2m - (2n - 2j + 1)) \qquad \text{by Property 4}$$

$$= \sum_{j=2}^{n} (2n - 2 - (2n - 2j + 1)) \qquad m = n - 1$$

$$= \sum_{j=2}^{n} (2j - 3)$$

$$= 3 - 3n + \sum_{j=2}^{n} 2j = 3 - 3n + 2 \sum_{j=2}^{n} j \qquad (n-1). - 3 = 3 - 3n$$

First attempt: a line L

L: p_1 — p_2 — ... — p_n with $n > 1$

L is p_1 linked to a pending line p_2, ..., p_n so previous properties apply with $i = 2$.

$$\begin{aligned}
\mathcal{H}_G(p_1, p_n) &= \sum_{j=2}^{n} \mathcal{H}_G(p_{j-1}, p_j) && \text{by Property 2} \\
&= \sum_{j=2}^{n} (2m - (2n - 2j + 1)) && \text{by Property 4} \\
&= \sum_{j=2}^{n} (2n - 2 - (2n - 2j + 1)) && m = n - 1 \\
&= \sum_{j=2}^{n} (2j - 3) \\
&= 3 - 3n + \sum_{j=2}^{n} 2j = 3 - 3n + 2 \sum_{j=2}^{n} j && (n-1). - 3 = 3 - 3n \\
&= 3 - 3n + 2\frac{(n+2)(n-1)}{2}
\end{aligned}$$

First attempt: a line L

L: p_1 — p_2 — ... — p_n with $n > 1$

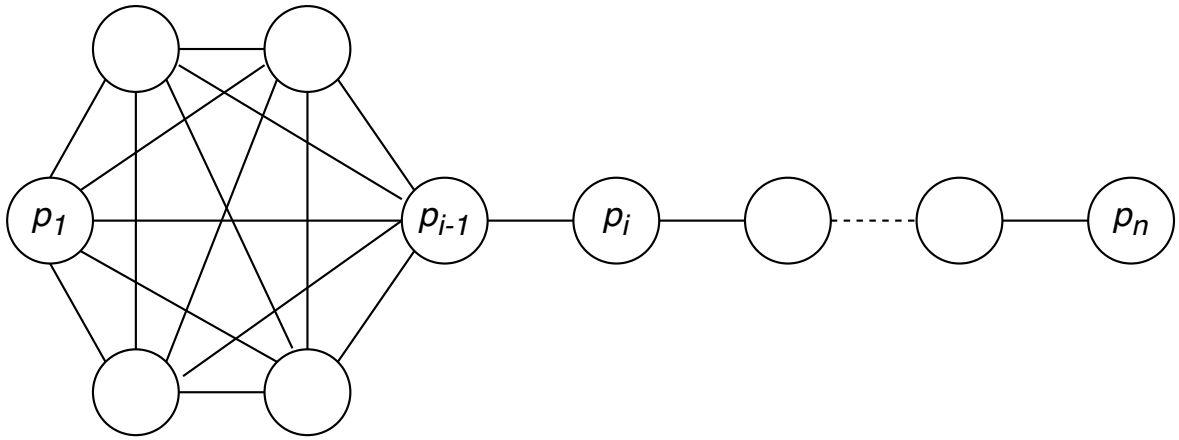
L is p_1 linked to a pending line p_2, ..., p_n so previous properties apply with $i = 2$.

$$\begin{aligned}
\mathcal{H}_G(p_1, p_n) &= \sum_{j=2}^{n} \mathcal{H}_G(p_{j-1}, p_j) && \text{by Property 2} \\
&= \sum_{j=2}^{n} (2m - (2n - 2j + 1)) && \text{by Property 4} \\
&= \sum_{j=2}^{n} (2n - 2 - (2n - 2j + 1)) && m = n - 1 \\
&= \sum_{j=2}^{n} (2j - 3) \\
&= 3 - 3n + \sum_{j=2}^{n} 2j = 3 - 3n + 2 \sum_{j=2}^{n} j && (n-1).-3 = 3 - 3n \\
&= 3 - 3n + 2\frac{(n+2)(n-1)}{2} \\
&= n^2 - 2n + 1 \in \Theta(n^2)
\end{aligned}$$

Second attempt: Lollipop

A lollipop consists of a clique linked by a bridge to a line

Let us consider a lollipop made of vertices $p_1, \ldots, p_n$ with $n > 2$ where p_1, p_{i-1} is the clique with $i > 2$ and a standard random walk starting from p_1



$$m = \frac{(i-1)(i-2)}{2} + n - (i-1) = \frac{i^2 - 5i}{2} + n + 2$$

Analysis (1/3)

Until reaching p_{i-1}, the probability of hitting p_{i-1} at the next step is $\frac{1}{i-2}$: it is a geometric law. Thus,

$$\mathcal{H}_G(p_1, p_{i-1}) = i - 2$$

We now compute $\mathcal{H}_G(p_1, p_n)$

Analysis (2/3)

$\mathcal{H}_G(p_1, p_n) =$

Analysis (2/3)

$$\mathcal{H}_G(p_1, p_n) = \mathcal{H}_G(p_1, p_{i-1}) + \sum_{j=i}^{n} \mathcal{H}_G(p_{j-1}, p_j) \qquad \text{by Property 2}$$

Analysis (2/3)

$$\mathcal{H}_G(p_1, p_n) = \mathcal{H}_G(p_1, p_{i-1}) + \sum_{j=i}^{n} \mathcal{H}_G(p_{j-1}, p_j) \qquad \text{by Property 2}$$

$$= i - 2 + \sum_{j=i}^{n} \mathcal{H}_G(p_{j-1}, p_j)$$

Analysis (2/3)

$$\mathcal{H}_G(p_1, p_n) = \mathcal{H}_G(p_1, p_{i-1}) + \sum_{j=i}^{n} \mathcal{H}_G(p_{j-1}, p_j) \qquad \text{by Property 2}$$

$$= i - 2 + \sum_{j=i}^{n} \mathcal{H}_G(p_{j-1}, p_j)$$

$$= i - 2 + \sum_{j=i}^{n} (2m - (2n - 2j + 1)) \qquad \text{by Property 4}$$

Analysis (2/3)

$$\mathcal{H}_G(p_1, p_n) = \mathcal{H}_G(p_1, p_{i-1}) + \sum_{j=i}^{n} \mathcal{H}_G(p_{j-1}, p_j) \qquad \text{by Property 2}$$

$$= i - 2 + \sum_{j=i}^{n} \mathcal{H}_G(p_{j-1}, p_j)$$

$$= i - 2 + \sum_{j=i}^{n} (2m - (2n - 2j + 1)) \qquad \text{by Property 4}$$

$$= i - 2 + (n - i + 1).(2m - 2n - 1) + 2\sum_{j=i}^{n} j$$

Analysis (2/3)

$$\mathcal{H}_G(p_1, p_n) = \mathcal{H}_G(p_1, p_{i-1}) + \sum_{j=i}^{n} \mathcal{H}_G(p_{j-1}, p_j) \qquad \text{by Property 2}$$

$$= i - 2 + \sum_{j=i}^{n} \mathcal{H}_G(p_{j-1}, p_j)$$

$$= i - 2 + \sum_{j=i}^{n} (2m - (2n - 2j + 1)) \qquad \text{by Property 4}$$

$$= i - 2 + (n - i + 1).(2m - 2n - 1) + 2\sum_{j=i}^{n} j$$

$$= i - 2 + (n - i + 1).(2m - n + i - 1)$$

Analysis (2/3)

$$\mathcal{H}_G(p_1, p_n) = \mathcal{H}_G(p_1, p_{i-1}) + \sum_{j=i}^{n} \mathcal{H}_G(p_{j-1}, p_j) \qquad \text{by Property 2}$$

$$= i - 2 + \sum_{j=i}^{n} \mathcal{H}_G(p_{j-1}, p_j)$$

$$= i - 2 + \sum_{j=i}^{n} (2m - (2n - 2j + 1)) \qquad \text{by Property 4}$$

$$= i - 2 + (n - i + 1).(2m - 2n - 1) + 2\sum_{j=i}^{n} j$$

$$= i - 2 + (n - i + 1).(2m - n + i - 1)$$

$$= i - 2 + (n - i + 1).(i^2 - 4i + n + 3)$$

Analysis (3/3)

Let $i := \frac{n}{4}$.

$$\begin{aligned}\mathcal{H}_G(p_1, p_n) &= \frac{n}{4} - 2 + (\frac{3n}{4} + 1).(\frac{n^2}{16} + 3) \\ &= \frac{3n^3}{64} + \frac{n^2}{16} + \frac{10n}{4} + 1 \in \Theta(n^3)\end{aligned}$$

Actually, the lollipop graph is shown to be the worst case in [6]: precisely the lollipops with a clique of $\frac{2n}{3}$ vertices

Roadmap

1. Introduction
2. Correctness
3. Complexity of the Standard Random Walk
   - Relevant Quantities
   - Tool: Markov Chains
   - Hitting Time of the Standard Random Walk
   - Cover Time of the Standard Random Walk
4. Optimal (Pure) Random Walk
5. Conclusion
6. References

Bounds

From [4, 3], we know that the cover time of the standard random walk is also in $\Theta(n^3)$.

Again, the worst-case graph is the lollipop with a clique of $\frac{2n}{3}$ vertices!

Interest of a bounded cover time

Simple Monte-Carlo Broadcast Algorithm

Let $C \geq C_G(P)$.

Assume u has a data d to broadcast.

Initialization

`deliver` d
pick $i \in \{1, \ldots, \delta_u\}$ according to P_u
send $\langle d, 1 \rangle$ via port number i

v **receives** $\langle d, i \rangle$

`deliver` d
if $i < C$ **then**
    pick $i \in \{1, \ldots, \delta_v\}$ according to P_v
    send $\langle d, i+1 \rangle$ via port number i
end if

Termination in C hops and partial correctness *w.h.p.* (works in anonymous networks; yet, duplicates ...).

Roadmap

1. Introduction
2. Correctness
3. Complexity of the Standard Random Walk
   - Relevant Quantities
   - Tool: Markov Chains
   - Hitting Time of the Standard Random Walk
   - Cover Time of the Standard Random Walk
4. Optimal (Pure) Random Walk
5. Conclusion
6. References

What is the issue with the standard random walk?

What is the issue with the standard random walk?

Lemma 4 claims that **the more the degree of a node is the more often it is visited!**

It is an issue!

Indeed

- In the lollipop, we have both very high degree nodes and very low degree nodes: the hitting time is in $\Theta(n^3)$
- In a line, degrees are almost equal (either 1 or 2): the hitting time is in $\Theta(n^2)$ although the diameter is maximal!

What is the issue with the standard random walk?

Lemma 4 claims that **the more the degree of a node is the more often it is visited!**

It is an issue!

Indeed

- In the lollipop, we have both very high degree nodes and very low degree nodes: the hitting time is in $\Theta(n^3)$
- In a line, degrees are almost equal (either 1 or 2): the hitting time is in $\Theta(n^2)$ although the diameter is maximal!

Solution: load balance the probability distributions

Probability Distributions proposed in [7]

$$p(u,v) = \begin{cases} \dfrac{\delta_v^{-1/2}}{\sum_{w \in N(u)} \delta_w^{-1/2}} & \text{if } v \in N(u) \\ 0 & \text{otherwise} \end{cases}$$

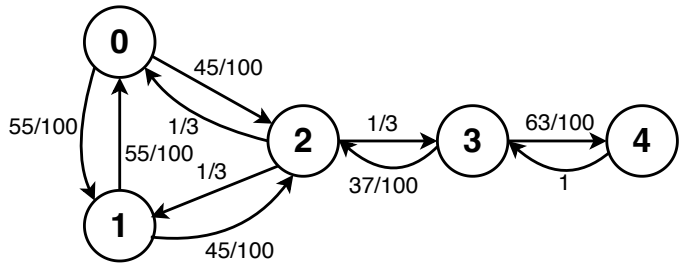
A minor drawback is that each node should know the degree of its neighbors

(but, it is still local information)

Probability Distributions proposed in [7]

$$p(u,v) = \begin{cases} \dfrac{\delta_v^{-1/2}}{\sum_{w \in N(u)} \delta_w^{-1/2}} & \text{if } v \in N(u) \\ 0 & \text{otherwise} \end{cases}$$

Markov chain of the random walk given in [7] on a lollipop



A minor drawback is that each node should know the degree of its neighbors

(but, it is still local information)

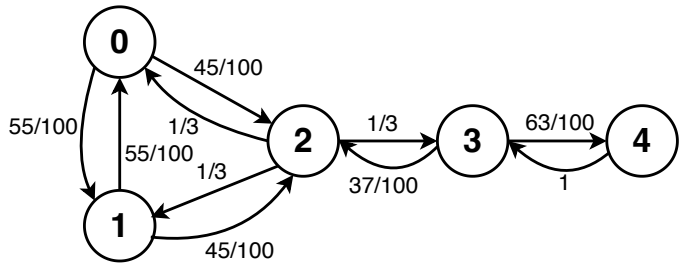
Probability Distributions proposed in [7]

$$p(u,v) = \begin{cases} \dfrac{\delta_v^{-1/2}}{\sum_{w \in N(u)} \delta_w^{-1/2}} & \text{if } v \in N(u) \\ 0 & \text{otherwise} \end{cases}$$

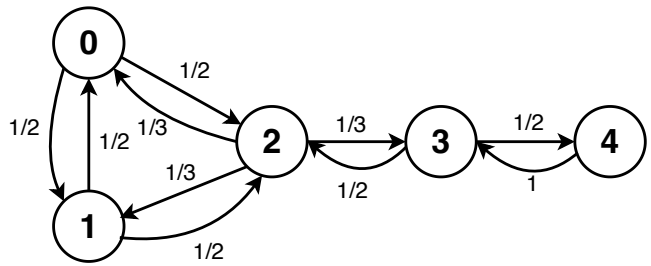
A minor drawback is that each node should know the degree of its neighbors

(but, it is still local information)

Markov chain of the random walk given in [7] on a lollipop



Markov chain of the standard random walk on the same graph



Bounds

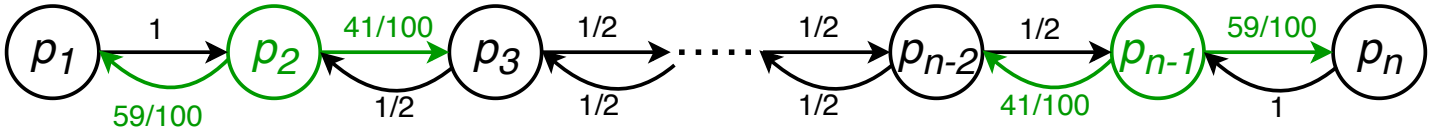
Hitting Time: $\Theta(n^2)$

It is the **optimal** distribution for the pure random walk

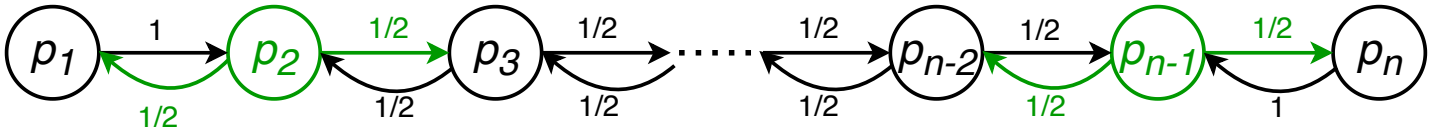
Cover Time: $O(n^2 \log n)$

A few more details

The lower bound is natural: in a line, only two vertices (p_2 and p_{n-1}) have distributions that differ from the standard random walk



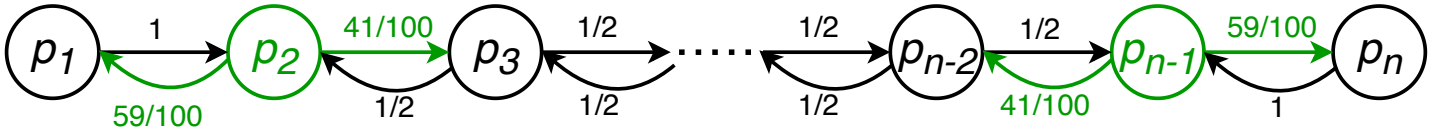
Markov chain of the random walk given in [7] on a line



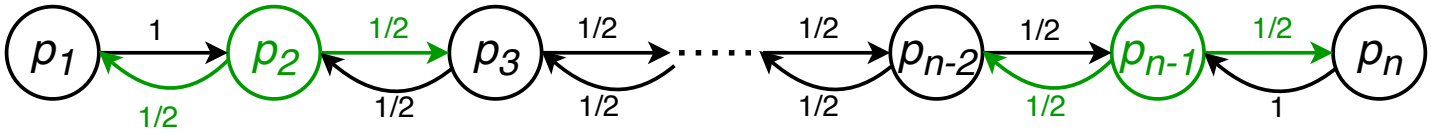
Markov chain of the standard random walk on a line

A few more details

The lower bound is natural: in a line, only two vertices (p_2 and p_{n-1}) have distributions that differ from the standard random walk



Markov chain of the random walk given in [7] on a line

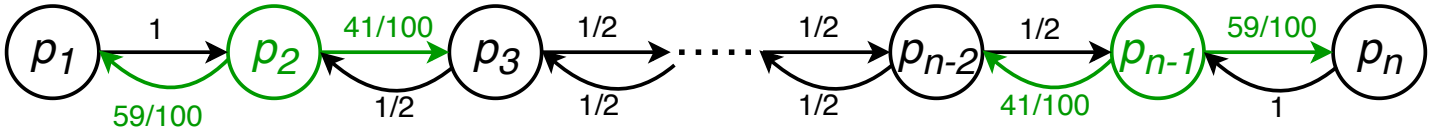


Markov chain of the standard random walk on a line

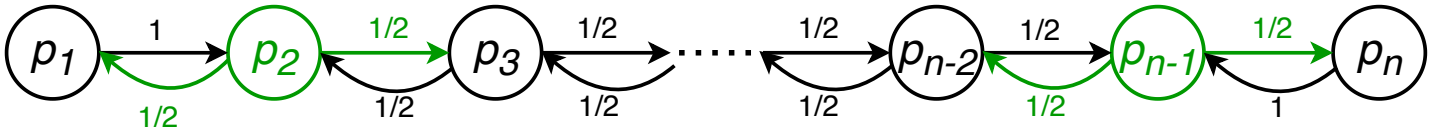
Intuition: With an arbitrary large line, the difference between the standard random walk and the one of [7] becomes negligible, thus we have $\Omega(n^2)$.

A few more details

The lower bound is natural: in a line, only two vertices (p_2 and p_{n-1}) have distributions that differ from the standard random walk



Markov chain of the random walk given in [7] on a line



Markov chain of the standard random walk on a line

Intuition: With an arbitrary large line, the difference between the standard random walk and the one of [7] becomes negligible, thus we have $\Omega(n^2)$.

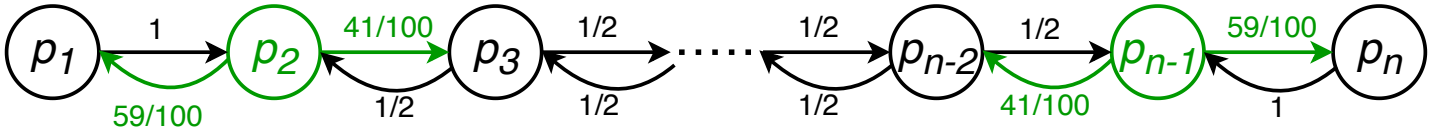
Proof: Assume G is a line $p_1 — \ldots — p_n$ with $n > 1$. Let $\mathfrak{H}_G(p_i, p_j)$ be the hitting time from p_i to p_j under the transition probability matrix of the random walk of [7].

$$\mathfrak{H}_G(p_1, p_n) > \mathfrak{H}_G(p_1, p_{n-1}) > \mathcal{H}_G(p_1, p_{n-1}) \in \Omega(n^2)$$

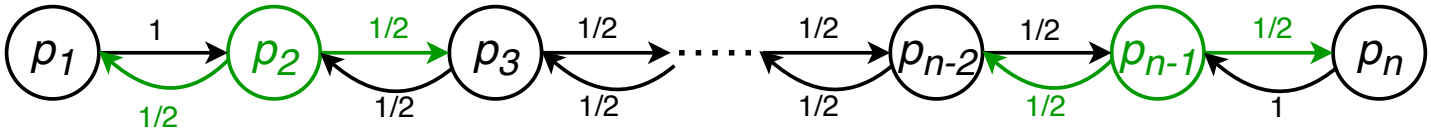
($n.b.$, $\mathfrak{H}_G(p_1, p_{n-1}) > \mathcal{H}_G(p_1, p_{n-1})$ since $\frac{41}{100} < \frac{1}{2}$) □

A few more details

The lower bound is natural: in a line, only two vertices (p_2 and p_{n-1}) have distributions that differ from the standard random walk



Markov chain of the random walk given in [7] on a line



Markov chain of the standard random walk on a line

Intuition: With an arbitrary large line, the difference between the standard random walk and the one of [7] becomes negligible, thus we have $\Omega(n^2)$.

Proof: Assume G is a line $p_1 — \ldots — p_n$ with $n > 1$. Let $\mathfrak{H}_G(p_i, p_j)$ be the hitting time from p_i to p_j under the transition probability matrix of the random walk of [7].

$$\mathfrak{H}_G(p_1, p_n) > \mathfrak{H}_G(p_1, p_{n-1}) > \mathcal{H}_G(p_1, p_{n-1}) \in \Omega(n^2)$$

($n.b.$, $\mathfrak{H}_G(p_1, p_{n-1}) > \mathcal{H}_G(p_1, p_{n-1})$ since $\frac{41}{100} < \frac{1}{2}$) □

The upper bound is more complex! (see [7])

Roadmap

1. Introduction
2. Correctness
3. Complexity of the Standard Random Walk
   - Relevant Quantities
   - Tool: Markov Chains
   - Hitting Time of the Standard Random Walk
   - Cover Time of the Standard Random Walk
4. Optimal (Pure) Random Walk
5. Conclusion
6. References

Pros and Cons of Random-walk-based Routing

Pros.

- Partially Correct
- Robust
- Adaptive
- Fair
- Messages: low message overhead and no control message
- Low memory at each process

Cons.

- Termination *almost sure* only
- Slow: $\Omega(n^2)$

  In many large-scale networks, the diameter is logarithmic in n, *e.g.*, IPv6, which allows for up to 2^{128} machines, assumes the diameter is at most 255!
- Not FIFO

Roadmap

1. Introduction
2. Correctness
3. Complexity of the Standard Random Walk
   - Relevant Quantities
   - Tool: Markov Chains
   - Hitting Time of the Standard Random Walk
   - Cover Time of the Standard Random Walk
4. Optimal (Pure) Random Walk
5. Conclusion
6. References

References

[1] G. R. Brightwell and P. Winkler.
Maximum hitting time for random walks on graphs.
Random Struct. Algorithms, 1(3):263–276, 1990.

[2] L. Euler.
Solutio problematis ad geometriam situs pertinentis.
Commentarii Academiae Scientiarum Imperialis Petropolitanae, 8:128–140, 1736.

[3] U. Feige.
A tight lower bound on the cover time for random walks on graphs.
Random Struct. Algorithms, 6(4):433–438, 1995.

[4] U. Feige.
A tight upper bound on the cover time for random walks on graphs.
Random Struct. Algorithms, 6(1):51–54, 1995.

[5] G. Frobenius.
Über Matrizen aus nicht negativen Elementen.
Preussische Akademie der Wissenschaften Berlin: Sitzungsberichte der Preußischen Akademie der Wissenschaften zu Berlin. Reichsdr., 1912.

[6] F. Gobel and A. Jagers.
Random walks on graphs.
Stochastic processes and their applications, 2(4):311–336, 1974.

[7] S. Ikeda, I. Kubo, and M. Yamashita.
The hitting and cover times of random walks on finite graphs using local degree information.
Theor. Comput. Sci., 410(1):94–100, 2009.

[8] O. Perron.
Zur theorie der matrices.
Mathematische Annalen, 64:248–263, 1907.